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ABSTRACT

This guide for teachers using the SMSG text materials for grade 5 considers four chapters on addition and subtraction of rational numbers, measurement of angles, area, and ratio, plus a review of the fifth-grade program. The objectives or purposes for each unit are given, followed by mathematical background. Detailed lesson plans are then provided, including sequences of statements and questions, activities, and exercise sets with answers. (MS)

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School Mathematics Study Group

Mathematics for the Elementary School, Grade 5

Unit 32

Mathematics for the Elementary School, Grade 5

Teacher's Commentary, Part II

REVISED EDITION

Prepared under the supervision of the
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Chapter 6

ADDITION AND SUBTRACTION OF RATIONAL NUMBERS

PURPOSE OF THE UNIT

The purposes of the unit are these:

1. To increase understanding of the meaning of rational numbers and their use as measures of regions, segments, and subsets of a set.

2. To develop understanding of fraction and decimal notation for rational numbers; to develop facility in renaming rational numbers by fraction and decimal numerals; to make use of complete factorizations of counting numbers for this purpose.

3. To develop understanding of the operations of addition and subtraction of rational numbers and of their properties.

4. To compute sums and addends using fraction and decimal numerals; to solve problems requiring the use of the operations of addition and subtraction.

5. To encourage pupils to discover relations and procedures for themselves.

MATHEMATICAL BACKGROUND

Introduction

In the study of mathematics in the elementary school, a child learns to use several sets of numbers. The first of these is the set of counting numbers, 1, 2, 3, 4, ... The second is the set of whole numbers, 0, 1, 2, 3, 4, ... The child also may have learned certain properties of whole numbers.

During the primary and middle grades the idea of "number" is enlarged, so that by the end of the sixth grade the child recognizes each of the following as a name for a number:

$$\frac{4}{3},\ \frac{1}{2},\ 3.6,\ 2\frac{1}{2},\ 8,\ 0,\ \frac{5}{5},\ \frac{6}{2},\ .01$$

In traditional language, we might say that when the child has completed the first six years of school mathematics he knows about "the whole numbers, fractions, decimals, and mixed numbers." This language is primarily numeral language. It obscures the fact that a single number can have names of many kinds. "Fractions, decimals, and mixed numbers" are kinds of number names rather than different kinds of numbers. Whether we make a piece of ribbon $1\frac{1}{2}$ in. long, or 1.5 in. long, or $\frac{3}{2}$ in. long makes no difference--our ribbon is the same whatever our choice of numeral. That is, $1\frac{1}{2}$, 1.5, $\frac{3}{2}$ are all names for the same number. This number is a member of a set of numbers sometimes called the non-negative numbers or the rational numbers of arithmetic. For our purposes here, we shall call them the rational numbers, realizing that they are only a subset of the set of all rational numbers. It also should be realized that within the set of rational numbers is a set which corresponds to the set of whole numbers. For example, 0, 3, 7 are all rational numbers that are also whole numbers. $\frac{3}{4}$, $\frac{7}{4}$, and .2 are rational numbers that are not whole numbers.

First Ideas About Rational Numbers

Children develop early ideas about rational numbers by working with regions--rectangular regions, circular regions, triangular regions, etc. In Figures A, B, and C, rectangular regions have been used. For any type of region we must first identify the unit region. In Figures A, B, and C, the unit region is a square region.

In Figures A and B we see that:

(1) The unit region has been separated into a number of congruent regions.

(2) Some of the regions have been shaded.

(a) *Using regions*. Let us see how children use regions to develop their first ideas of rational numbers. The child learns in simple cases to associate a number like $\frac{1}{2}$ or $\frac{2}{3}$ with a shaded portion of the figure. (Rational numbers can also be associated with the unshaded portions.)

Using two or more congruent regions (Fig. C), he can separate each into the same number of congruent parts and shade some of the parts. Again, he can associate a number with the resulting shaded region.

Fig. A

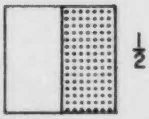


Fig. B

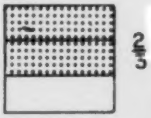
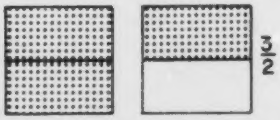


Fig. C



The unit square is separated into 2 congruent regions. 1 is shaded.

The unit square is separated into 3 congruent regions. 2 are shaded.

Each unit square is separated into 2 congruent regions. 3 are shaded. We have $\frac{3}{2}$ of a *unit square*.

At this point, the child is only at the beginning of his concept of rational numbers. However, let us note what we are doing when we introduce, for example $\frac{2}{3}$. We separate the (unit) region into 3 congruent parts. Then we shade 2 of these parts. Similarly, in $\frac{3}{2}$, we separate each (unit) region into 2 congruent regions, and shade 3 parts. In using regions to represent a number like $\frac{3}{2}$, we must emphasize the fact that we are thinking of $\frac{3}{2}$ of a unit region, as in Fig. C.

(b) Using the number line. The steps used with regions can be carried out on the number line. It is easy to see that this is a very practical thing to do. If we have a ruler marked only in inches, we cannot make certain types of useful measurements. We need to have points between the unit intervals, and we would like to have numbers associated with these points.

The way we locate new points on the ruler parallels the procedure we followed with regions. We mark off each unit segment into congruent parts. We count off these parts. Thus, in order to locate the point corresponding to $\frac{2}{3}$, we must mark off the unit segment in 3 congruent parts. We then count off 2 of them. (Fig. D) If we have separated each unit interval in 2 congruent parts and counted off 3 of them, we have located the point which we would associate with $\frac{3}{2}$. (Fig. E)

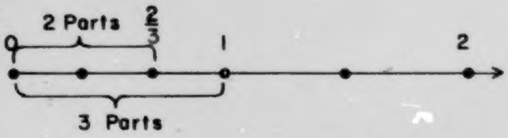


Fig. D

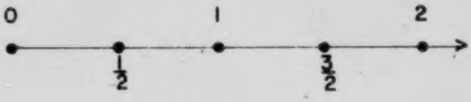


Fig. E

Once we have this construction in mind, we see that all such numbers as $\frac{3}{4}$, $\frac{5}{8}$, $\frac{2}{4}$, $\frac{1}{3}$, $\frac{4}{3}$, $\frac{11}{8}$ can be associated with particular points on the number line. To locate $\frac{11}{8}$, for example, we mark the unit segments into 8 congruent segments. (Fig. F)

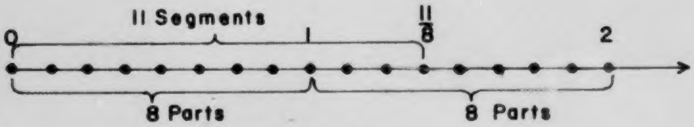


Fig. F

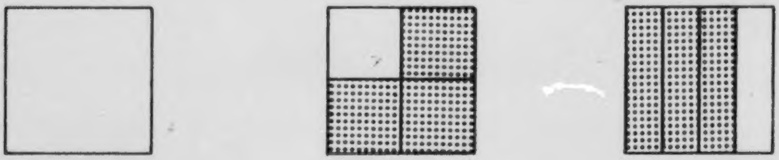
(c) Numerals for pairs of numbers. Suppose that we consider a pair of counting numbers such as 11 and 8 where 11 is the first number and 8 is the second number. We can make a symbol, writing the name of the first number of the pair above the line and that of the second below. Thus for the pair of numbers, 11 and 8, our symbol would be $\frac{11}{8}$. If we had thought of 8 as the first number of the pair and 11 as the second, we would have said the pair 8 and 11, and the symbol would have been $\frac{8}{11}$. For the numbers 3 and 4, the symbol would be $\frac{3}{4}$. For the numbers 4 and 3, the symbol would be $\frac{4}{3}$.

With the symbol described in the preceding paragraph, we can associate a point on the number line. The second number tells into how many congruent segments to separate each unit segment. The first number tells how many segments to count off.

We also can associate each of our symbols with a shaded region as in Fig. A, B, and C. The second number tells us into how many congruent parts we must separate each unit region. The first number tells us how many of these parts to shade.

For young children, regions are easier to see and to work with than segments. However, the number line has one strong advantage. For example, we associate a number as $\frac{3}{4}$, with exactly one point on the number line. The number line also gives an unambiguous picture for numbers like $\frac{3}{2}$ and $\frac{7}{2}$. A region corresponding to $\frac{3}{4}$ is less precisely defined in that regions with the same measure need not be identical or even congruent.

In Fig. G, we can see that each shaded region is $\frac{3}{4}$ of a unit square. Recognizing that both shaded regions have $\frac{3}{4}$ sq. units is indeed one part of the area concept.



Unit square

Figure G

When we match numbers with points on the number line, we work with segments that begin at 0. For this reason, though the number line is less intuitive at early stages, it is well to use it as soon as possible.

Meaning of Rational Number

The diagrams Fig. H, (a), (b), (c), show a number line on which we have located points corresponding to $\frac{1}{2}$, $\frac{2}{2}$, $\frac{3}{2}$, etc. and a number line on which we have located points corresponding to $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, etc. Also shown, is a number line with $\frac{1}{8}$, $\frac{2}{8}$, etc. As we look at these lines, we see that it seems very natural to think of $\frac{0}{2}$ as being associated with the 0 point. We are really, so to speak, counting off 0 segments. Similarly, it seems natural to locate $\frac{0}{4}$ and $\frac{0}{8}$ as indicated.

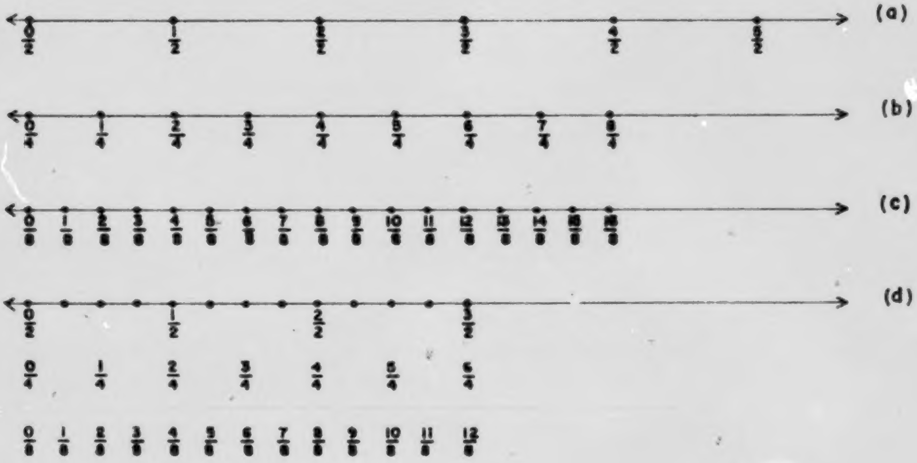

(a)
0/2 1/2 2/2 3/2 4/2 5/2
(b)
0/4 1/4 2/4 3/4 4/4 5/4 6/4 7/4 8/4
(c)
0/8 1/8 2/8 3/8 4/8 5/8 6/8 7/8 8/8 9/8 10/8 11/8 12/8 13/8 14/8 15/8 16/8
(d)
0/2 1/2 2/2 3/2
0/4 1/4 2/4 3/4 4/4 5/4 6/4
0/8 1/8 2/8 3/8 4/8 5/8 6/8 7/8 8/8 9/8 10/8 11/8 12/8

Figure H

Now let us put our diagrams (a), (b), (c) together. In other words, let us carry out on a single line (d) the process for locating all the points.

When we do this, we see that $\frac{1}{2}$, $\frac{2}{4}$, and $\frac{4}{8}$ are all associated with the same point. In the same way, $\frac{3}{4}$ and $\frac{6}{8}$ are associated with the same point.

Now we are ready to explain more precisely what we mean by fraction and by rational number. Let us agree to call the symbols we have been using fractions. A fraction, then, is a symbol associated with a pair of numbers. The first number of the pair is called the numerator and the second number is called the denominator. So far, we have used only those fractions in which the numerator of the number pair is a whole number (0, 1, 2, ...), and the denominator is a counting number (1, 2, 3, ...).

Each fraction can be used to locate a point on the number line. To each point located by a fraction there corresponds a rational number. Thus, a fraction names the rational number. For example, if we are told the fraction $\frac{3}{10}$, we can locate a point that corresponds to it on the number line. $\frac{3}{10}$ is the name of the rational number associated with this point. This point, however, can also be located by means of other fractions, such as $\frac{6}{20}$ and $\frac{9}{30}$. Thus, $\frac{6}{20}$ and $\frac{9}{30}$ also are names for the rational number named by $\frac{3}{10}$ since they are associated with the same point. Rational numbers, then, are named by fractions of the type we have been discussing. To each point on the number line that can be located by a fraction, there corresponds a non-negative rational number.

A very unusual child might wonder whether every point on the number line can be located by a fraction of the kind we have described. We must answer "No". There are numbers--π being one of them and $\sqrt{2}$ being another--that have no fraction names of the sort we have described. Introducing such irrational numbers is deferred until the seventh and eighth grades.

The Whole Numbers As Rational Numbers

Our pattern for matching fractions with points on the number line can be used with these fractions: $\frac{0}{1}$, $\frac{1}{1}$, $\frac{2}{1}$, $\frac{3}{1}$, etc.

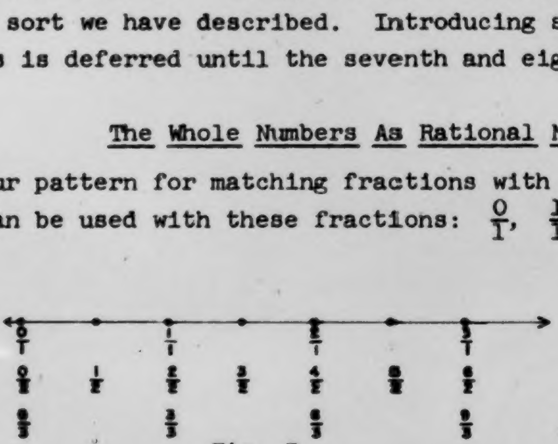


Fig. I

On the number line we see (Fig. I) that we matched $\frac{1}{1}$, $\frac{2}{2}$, $\frac{3}{3}$ with the same point. We note that this point is also matched with the counting number 1. Thus, to the same point corresponds

(1) the counting number 1

(2) the rational number named by $\frac{1}{1}$.

It seems that it would be a convenience to use the symbol 1 as still another name for the rational number named by $\frac{1}{1}$, $\frac{2}{2}$, etc. This would allow us to write $1 = \frac{2}{2}$, for example. In the same way, we would think of 5 as another name for the number named by $\frac{5}{1}$, $\frac{10}{2}$, etc.

We need at this point to be a little careful in our thinking. There is nothing illogical about using any symbol we like as a numeral. A problem *does* arise, however, when a single symbol has *two* meanings, because then we are in obvious danger that inconsistencies may result. For example, when we think of 2, 3, and 6 as *counting* numbers we are accustomed to writing $2 \times 3 = 6$. We will eventually define the product of two rational numbers, and we would be in serious trouble if the product of the *rational* numbers named by 2 and 3 were anything but the *rational* number named by 6.

However, using 0, 1, 2, 3, etc., as names for rational numbers never leads us into any inconsistency. For all the purposes of arithmetic--that is, for finding sums, products, etc., and for comparing sizes, we get names for whole numbers or names for rational numbers. In more sophisticated mathematical terms, we can say that the set of rational numbers contains a subset--those named by $\frac{0}{1}$, $\frac{1}{1}$, $\frac{2}{1}$, etc.--isomorphic to the set of whole numbers, that is $\frac{0}{1}$, $\frac{1}{1}$, etc. behave just like whole numbers, 0, 1, etc.

It would be overambitious to attempt to formulate the idea of isomorphism precisely in our teaching. It is sufficient for our purposes to regard 0, 1, 2, etc., as names for rational numbers. It is appropriate to note, however, in connection with operations on rationals, that where the operations are applied to numbers like $\frac{1}{1}$, $\frac{2}{1}$ they lead to results already known from experience with whole numbers.

Identifying Fractions That Name The Same Rational Number

When we write $\frac{1}{2} = \frac{3}{6}$, we are saying "$\frac{1}{2}$ and $\frac{3}{6}$ are names for the same number."

(a) Using physical models. The truth of the sentence $\frac{1}{2} = \frac{3}{6}$ can be discovered by concrete experience. In Fig. J, for example, we have first separated our unit region into two congruent regions. We have then separated each of these parts further into 3 congruent regions as shown in the second drawing. The second unit square is thus separated into 2 × 3, or 6 parts. Shading 1 part in the first drawing is equivalent to shading 1 × 3, or 3 parts in the second. We thus recognize that $\frac{1}{2} = \frac{1 \times 3}{2 \times 3}$.

Shading $\frac{1}{2}$ and $\frac{3}{6}$ of a region.

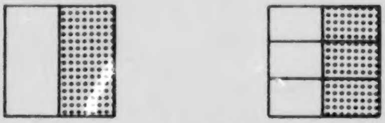


Fig. J

Again, our analysis of regions follows a pattern that can be applied on the number line. Let us consider $\frac{1}{2}$ and $\frac{4}{8}$.

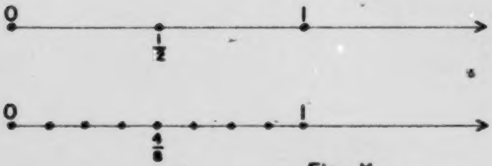


Fig. K

In locating $\frac{1}{2}$ on the number line, (Fig. K) we separate the unit interval into 2 congruent segments. In locating $\frac{4}{8}$, we separate it into 8 congruent segments. We can do this by first separating into 2 parts and then separating each of these 2 segments into 4 segments. This process yields (2 × 4) congruent segments. Taking 1 of 2 congruent parts thus leads to the same point as taking 4 of 8 congruent parts:

$$\frac{1}{2} = \frac{1 \times 4}{2 \times 4}$$

In other words, when we multiply the numerator and denominator of $\frac{1}{2}$ by the same counting number, we can visualize the result using the number line. We have subdivided our $\frac{1}{2}$ intervals into a number of congruent parts.

After many such experiences, children should be able to make a picture to explain this type of relationship. For example, region and number line pictures for $\frac{3}{4} = \frac{3 \times 2}{4 \times 2}$ are shown in Fig. L.

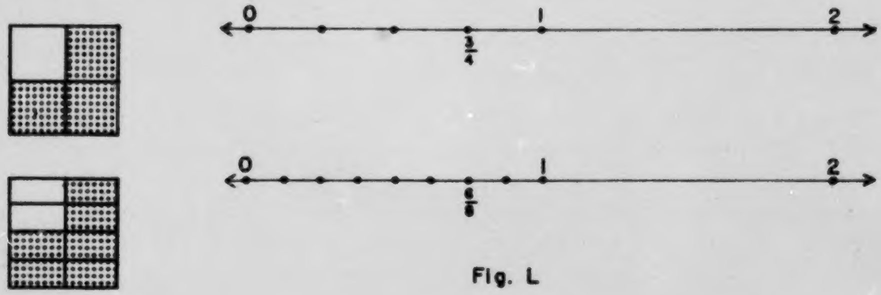


Fig. L

Each $\frac{1}{4}$ part (region or interval) is subdivided into 2 congruent parts; hence $\frac{3}{4} = \frac{3 \times 2}{4 \times 2}$.

(b) Using numerators and denominators. In a discussion about two fractions naming the same number, it may appear startling to emphasize multiplying numerator and denominator by the same counting number. We usually think about finding the simplest fraction name if we can. We think, then, $\frac{4}{8} = \frac{1}{2}$. But, of course "=" means "names the same number." Seeing $\frac{1}{2} = \frac{4}{8}$, we can think, $\frac{4}{8} = \frac{1}{2}$, and this will be particularly easy if the "names the same number" idea has been emphasized adequately.

Another familiar idea also is contained in what has been said. We often think about dividing numerator and denominator by the same counting number. For example, we think:

$$\frac{6}{8} = \frac{6 \div 2}{8 \div 2} = \frac{3}{4}$$

This is easy to translate into a multiplicative statement, since multiplication and division are inverse operations: $6 \div 2 = 3$ means $3 \times 2 = 6$.

(c) Using factoring. The idea that multiplying the numerator and denominator of a fraction by a counting number gives a new fraction that names the same number as the original fraction is an idea very well suited to the discussion in the unit on factoring. To find a simpler name for $\frac{12}{15}$, we write:

$$\frac{12}{15} = \frac{2 \times 2 \times 3}{5 \times 3} = \frac{2 \times 2}{5} = \frac{4}{5}$$

Suppose we are thinking about two fractions. How will we decide whether or not they name the same number? There are two possibilities.

Rule (1). It may be that for such fractions as $\frac{1}{2}$ and $\frac{2}{4}$, one fraction is obtained by multiplying the numerator and denominator of the other by a counting number. In other words, it may be that we can picture the fractions as was just done. Since $\frac{2}{4} = \frac{2 \times 1}{2 \times 2}$, $\frac{2}{4}$ and $\frac{1}{2}$ belong to the same set--thus name the same number.

Rule (2). It may be that, we cannot use Rule 1 directly. For example, $\frac{2}{4}$ and $\frac{3}{6}$ cannot be compared directly by Rule 1. However, we can use Rule 1 to see that $\frac{2}{4} = \frac{1}{2}$ and $\frac{3}{6} = \frac{1}{2}$, and in this way, we see that $\frac{2}{4}$ and $\frac{3}{6}$ name the same number.

Notice that in comparing $\frac{2}{4}$ and $\frac{3}{6}$, we might have used Rule 1 and 2 in a different way. We might have recognized that:

$$\frac{2}{4} = \frac{2 \times 3}{4 \times 3} = \frac{6}{12} \qquad \text{and} \qquad \frac{3}{6} = \frac{3 \times 2}{6 \times 2} = \frac{6}{12}.$$

or we might have said:

$$\frac{2}{4} = \frac{2 \times 6}{4 \times 6} = \frac{12}{24} \qquad \text{and} \qquad \frac{3}{6} = \frac{3 \times 4}{6 \times 4} = \frac{12}{24}.$$

In the latter example, we have renamed $\frac{2}{4}$ and $\frac{3}{6}$, using fractions with denominator 4×6. Of course, we recognize that $4 \times 6 = 6 \times 4$. (Commutative Property)

In our example, we see that 24 is a common denominator for $\frac{2}{4}$ and $\frac{3}{6}$, though it is not the least common denominator. Nevertheless, one common denominator for two fractions is always the product of the two denominators.

(d) *A special test*. Let us now consider a special test for two fractions that name the same rational number. In our last example we used 6×4 as the common denominator for $\frac{2}{4}$ and $\frac{3}{6}$. Thus we had

$$\frac{2}{4} = \frac{2 \times 6}{4 \times 6} \qquad \text{and} \qquad \frac{3}{6} = \frac{3 \times 4}{6 \times 4}.$$

We could say: It is true that $\frac{2}{4} = \frac{3}{6}$, because the two resulting numerators--2×6 and 3×4--are equal, and the denominators are equal.

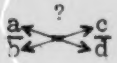
In other words, to test whether $\frac{2}{4} = \frac{3}{6}$, it is only necessary--*once you have understood the reasoning*--to test whether $2 \times 6 = 3 \times 4$. And this last number sentence is true!

In the same way, we can test whether $\frac{9}{15} = \frac{24}{40}$ by testing whether $9 \times 40 = 8 \times 15$. They do! When we do this, we are thinking:

$$\frac{9}{15} = \frac{9 \times 40}{15 \times 40} \qquad \text{and} \qquad \frac{24}{40} = \frac{24 \times 15}{40 \times 15}$$

This is an example of what is sometimes called "cross product rule." It is very useful in solving proportions. (Sometimes it is stated: The product of the means equals the product of the extremes.)

The rule states: To test whether two fractions $\frac{a}{b}$ and $\frac{c}{d}$ name the same number, we need only test whether $a \times d = b \times c$. That is,



This rule is important for later applications in mathematics such as similar triangles. In advanced texts on algebra, it is sometimes used as a way of *defining* rational numbers. That is, an advanced text might say: "A rational number is a set of symbols like $\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots\}$. Two symbols, $\frac{a}{b}$ and $\frac{c}{d}$, belong to the same set if $a \times d = b \times c$."

What we have done amounts to the same thing, but is developed more intuitively. For teaching purposes, the "multiply numerator and denominator by the same counting number" idea conveyed by Rule 1 can be visualized more easily than can the "cross product" rule.

It would certainly not be our intention to insist that children learn Rules 1 and 2 formally. However, these rules summarize an experience that is appropriate for children. We can form a chain of fractions that name the same number,

$$\frac{1}{2} = \frac{2}{4} = \frac{4}{8} = \frac{8}{16} = \cdots$$

Each fraction is formed by multiplying the numerator and denominator of the preceding one by 2. We can visualize this as subdividing repeatedly a segment or a region. (Rule 1). We can form a second chain beginning with $\frac{1}{2} = \frac{3}{6} = \frac{9}{18} \cdots$ We can then understand that it is possible to pick out any numeral from one chain and equate it with any numeral from the other, which is just what Rule 2 says.

Meaning of Rational Number - Summary

Let us summarize how far we have progressed in our development of the rational numbers.

(1) We regard a symbol like one of the following as naming a rational number:

$$\frac{3}{8},\ \frac{0}{5},\ \frac{7}{6},\ 6,\ \frac{4}{3},\ \frac{6}{4},\ 1,\ \frac{5}{5}$$

(2) We know how to associate each such symbol with a point on the number line.

(3) We know that the same rational number may have many names that are fractions. Thus, $\frac{6}{4}$ and $\frac{3}{2}$ are fraction names for the same number.

(4) We know that when we have a rational number named by a fraction, we can multiply the numerator and denominator of the fraction by the same counting number to obtain a new fraction name for the same rational number.

(5) We know that in comparing two rational numbers it is useful to use fraction names that have the same denominators. We know, too, that for any two rational numbers, we can always find fraction names of this sort.

Thus far we have not stressed what is often called, in traditional language, "reducing fractions." to "reduce" $\frac{6}{8}$, for example, is simply to name it with the name using the smallest possible numbers for the numerator and the denominator. Since 2 is a factor both of 6 and 8, we see that

$$\frac{6}{8} = \frac{3 \times 2}{4 \times 2} = \frac{3}{4}$$

We have applied our general idea that "multiplying numerator and denominator by the same counting number" gives a new name for the same number. We can call $\frac{3}{4}$ the simplest name for the rational number it names.

We would say that we have found, in $\frac{3}{4}$, the simplest name for the rational number named by $\frac{6}{8}$. This is more precise than saying we have "reduced" $\frac{6}{8}$, since we have not made the rational number named by $\frac{6}{8}$ any smaller. We have used another pair of numbers to rename it.

(6) We know, also, that 2 and $\frac{2}{1}$ name the same number. We thus regard the set of whole numbers as a subset of the set of rational numbers. Any number in this subset has a fraction name with denominator 1. ($\frac{0}{1}$, $\frac{1}{1}$, $\frac{2}{1}$, etc. belong to this subset.) 2 is a name for a rational number which is a whole number. 2 is not a fraction name for this number, but the number has fraction names $\frac{2}{1}$, $\frac{4}{2}$, etc.

At this point, it seems reasonable to use "number" for rational numbers where the meaning is clear. We may ask for the number of inches or measure of a stick, or the number of hours in a school day.

(7) We can agree to speak of the number $\frac{2}{3}$, to avoid the wordiness of "number named by $\frac{2}{3}$." Thus, we might say that the number $\frac{2}{3}$ is greater than the number $\frac{1}{2}$ (as we can verify easily on the number line). This would be preferable to saying that "the fraction $\frac{2}{3}$ is greater than the fraction $\frac{1}{2}$," because we do not mean that one name is greater than another.

(8) We should not say that 3 is the denominator of the number $\frac{2}{3}$, because the same number has other names (like $\frac{4}{6}$) with different denominators. 3 is rather the denominator of the fraction $\frac{2}{3}$.

(9) We have seen that the idea of rational number is relevant both to regions and line segments. We will see soon how it relates to certain problems involving sets.

Now we might introduce some decimals. The numeral, .1, for example, is another name for $\frac{1}{10}$. However, we can explain a numeral like 1.7 more easily when we have developed the idea of adding rational numbers.

Operations on Rational Numbers

Now let us consider the operations of arithmetic for rational numbers. For each, our treatment will be based on three considerations:

(1) The idea of rational number grows out of ideas about regions and the number line. Similarly, each operation on rational numbers can be "visualized" in terms of regions or the number line. Indeed, this is how people originally formed the ideas of sum, product, etc. of rational numbers. Each operation was introduced to fit a useful physical situation and not as a way of supplying problems for arithmetic textbooks.

(2) We recall that some rational numbers are whole numbers. So, we want our rules of operation to be consistent with what we already know about whole numbers.

(3) We must remember that the same rational number has many names. We will want to be sure that the result of an operation on two numbers does not depend on the special names we choose for them. For example, we want the sum of $\frac{1}{2}$ and $\frac{1}{3}$, to be the same number as the sum of $\frac{2}{4}$ and $\frac{2}{6}$.

These three ideas will guide us in defining the operations of addition, subtraction, multiplication, and division of rational numbers.

Addition and Subtraction

As an illustration of addition, we might think of a road by which stand a house, a school and a store as shown in Fig. M. If it is $\frac{1}{2}$ mile from the house to the school, and $\frac{1}{4}$ mile from the school to the store, then we can see that the distance from the house to the store is $\frac{3}{4}$ mile.

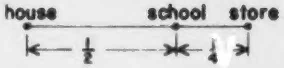


Fig. M

From such examples we can see the utility of defining addition of rational numbers by using the number line. To find the sum of $\frac{2}{5}$ and $\frac{4}{5}$ we would proceed as in Figure N.

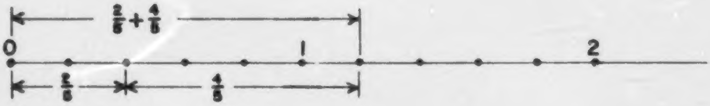


Fig. N

Using a ruler, we can locate the point on the number line corresponding to the sum of any two rational numbers. For example, with appropriate rulers, a child can locate the point that corresponds to the sum of $\frac{4}{5}$ and $\frac{3}{8}$. But a child would also like to know that the point for the sum located with a ruler is one for which he can find a fraction name--a fraction that names a rational number. Of course, one name for the sum of $\frac{4}{5}$ and $\frac{3}{8}$ is $\frac{4}{5} + \frac{3}{8}$, but what is the single fraction that names this number? Also, he is interested in knowing whether or not the set of rational numbers is closed under addition, since he knows that this is true for the whole numbers.

Using the number line, it is evident that the sum of $\frac{2}{4}$ and $\frac{3}{4}$ is $\frac{5}{4}$. This suggests a way to find the sum of two rational numbers that are named by fractions with the same denominator. For such fractions, we simply add the numerators. Thus, $\frac{3}{8} + \frac{4}{8} = \frac{3 + 4}{8} = \frac{7}{8}$. This definition matches the idea of joining two line segments.

But we are not finished! For suppose that we want to add $\frac{2}{3}$ and $\frac{1}{4}$. We know that there are many other names for the number named by $\frac{2}{3}$. Some are:

$$\frac{4}{6}, \frac{6}{9}, \frac{8}{12}.$$

Likewise, there are many names for the number named by $\frac{1}{4}$. They include:

$$\frac{2}{8}, \frac{3}{12}, \frac{4}{16}.$$

In order to find the sum of these two rational numbers we simply look for a pair of names with the same denominator--that is, with a common denominator. Having found them, we apply our simple process of adding numerators.

Thus we can write a fraction name for the sum of two rational numbers if we can write fraction names with the same denominators for the numbers. This we can always do, for to find the common denominator of two fractions, we need only to find the product of their denominators.

This provides a good argument as to why

$$\frac{2}{3} + \frac{1}{4} = \frac{11}{12}.$$

It is clear that $\frac{8}{12} + \frac{3}{12} = \frac{11}{12}$. If the idea that the same number has many names makes any sense at all, it must be true that

$$\frac{2}{3} + \frac{1}{4} = \frac{8}{12} + \frac{3}{12}.$$

Suppose that in our example we had used a different common denominator, as 24. Would we get a different result? We see that we would not for:

$$\begin{array}{ccccc}
\frac{2}{3} & = & \frac{2 \times 8}{3 \times 8} & = & \frac{16}{24} \\
\frac{1}{4} & = & \frac{1 \times 6}{4 \times 6} & = & \frac{6}{24} \\
\hline
\frac{2}{3} + \frac{1}{4} & & & = & \frac{22}{24} \\
\text{and } \frac{22}{24} & & & = & \frac{11}{12}.
\end{array}$$

Little needs be said here about subtraction. Using the number line we can visualize $\frac{3}{4} - \frac{2}{3}$ as in Figure 0.

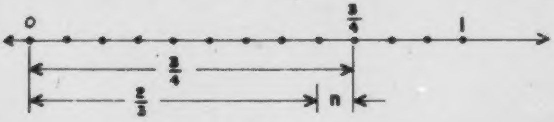


Fig. 0

Thus we can define $\frac{3}{4} - \frac{2}{3}$ as the number n such that $\frac{2}{3} + n = \frac{3}{4}$. Again, skillfully chosen names lead at once to the solution:

$$\frac{8}{12} + n = \frac{9}{12}$$

$$n = \frac{1}{12}$$

Properties of Addition for Rational Numbers

Our rule for adding rational numbers has some by-products worth noting.

We can see, for one thing, that addition of rational numbers is commutative. Our number line diagram illustrates this. In Fig. P we see the diagram for $\frac{1}{5} + \frac{2}{5}$ and for $\frac{2}{5} + \frac{1}{5}$.

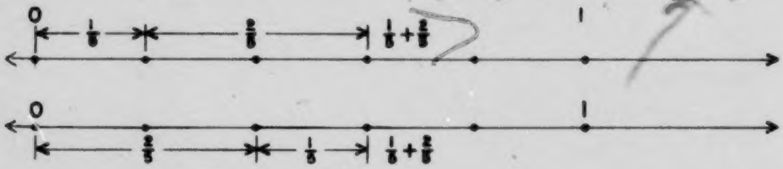


Fig. P

The commutative property also can be explained in another way.

$$\frac{1}{5} + \frac{2}{5} = \frac{1+2}{5} \quad \text{and} \quad \frac{2}{5} + \frac{1}{5} = \frac{2+1}{5}$$

We know that $1 + 2 = 2 + 1$, so we see that $\frac{1}{5} + \frac{2}{5} = \frac{2}{5} + \frac{1}{5}$. In general, to add rational numbers named by fractions with the same denominator we simply add numerators. Adding numerators involves adding whole numbers. We know that addition of whole numbers is commutative. This leads us to conclude that addition of rational numbers is also commutative.

We can use this type of discussion or the number line diagram to see that addition of rational numbers is also associative.

Here is another interesting property of addition of rational numbers. We recall that $\frac{0}{1}$, $\frac{0}{2}$, $\frac{0}{3}$, etc. are all names for 0. Thus $0 + \frac{3}{4} = \frac{0}{4} + \frac{3}{4} = \frac{0+3}{4} = \frac{3}{4}$. In general we see that the sum of 0 and any rational number is the number.

Similarly we recall, for example, that $\frac{2}{1}$ and $\frac{3}{1}$ are fraction names for 2 and [illegible] respectively. Thus

$$2 + 3 = \frac{2}{1} + \frac{3}{1} = \frac{2 + 3}{1} = \frac{5}{1} = 5$$

as we would expect (and hope).

Addition of rational numbers is not difficult to understand, once the idea that the same rational number has many different fraction names has been well established. The technique of computing sums of rational numbers written with fraction names is in essence a matter of finding common denominators. This is essentially the problem of the least common multiple and thus is a problem about whole numbers.

Addition of Rational Numbers Using Other Numerals

Often it is convenient to use numerals other than fractions to find the sum of two rational numbers. Those commonly used are mixed forms and decimals.

The first kind of numeral can be easily understood once addition has been explained. We can see with line segments that

$$2 + \frac{1}{3}$$

is a rational number, and it is also easy to see that $\frac{7}{3}$ is another name for this same number. Similarly, $\frac{15}{4} = 3\frac{3}{4}$. Indeed, these ideas can be introduced before any formal mechanism for adding two rational numbers named by fractions has been developed, because the idea that $2 + \frac{1}{3} = \frac{7}{3}$ goes back to the number line idea of sum. To adopt the convention of writing $2\frac{1}{3}$ as an abbreviation for $2 + \frac{1}{3}$ is then easy, and we may use a numeral like $2\frac{1}{3}$ as a name for a rational number. It is these we call a numeral in mixed form.

The use of decimals is still another convention for naming rational numbers. For example, 3.2 names a rational number; other names for this number are

$$3 + \frac{2}{10},\quad 3\frac{2}{10},\quad 3\frac{1}{5},\quad \frac{16}{5},\quad \frac{32}{10},\quad \frac{320}{100}.$$

Of these, $\frac{16}{5}$, $\frac{32}{10}$, and $\frac{320}{100}$ are fraction names while $3\frac{2}{10}$ and $3\frac{1}{5}$ are mixed forms.

The methods for computing with decimals are direct outcomes of their meaning. For example, to compute $3.4 + 1.7$, we may proceed as follows:

$$\begin{array}{rl} 3.4 = & 3 + \frac{4}{10} \\ 1.7 = & 1 + \frac{7}{10} \\ \hline & 4 + \frac{11}{10} = 5 + \frac{1}{10}. \end{array}$$

Hence $3.4 + 1.7 = 5.1$.

We want the child to develop a more efficient short-cut procedure for finding such a sum. However, the understanding of the procedure can be carried back, as shown, to the knowledge he already has about adding numbers with names in fraction or mixed form.

In a similar way, the procedures for subtracting, multiplying and dividing numbers named by decimals can be understood in terms of the same operations applied to numbers named by fractions.

Multiplication

By the time the child is ready to find the product of two rational numbers such as $\frac{2}{3}$ and $\frac{3}{4}$, he has already had a number of experiences in understanding and computing products of whole numbers.

He has seen 3×2 in terms of a rectangular array. He can recognize the arrangement in Fig. Q as showing 3 groups of objects with 2 objects in each group.

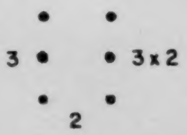


Fig. Q

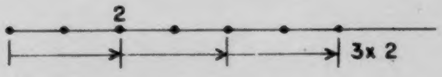


Fig. R

Also, he has seen 3×2 in terms of line segments. (Fig. R) that is, as a union of 3 two-unit segments. Furthermore, he has interpreted

Figure R in terms of travel along a line. For example, if he rows a boat 2 miles an hour across a lake, then in 3 hours he rows 6 miles.

Finally, 3 × 2 can be related to areas as in Figure S. Thus, a child has seen that the operation of multiplication can be applied to many physical models. He has related several physical situations to a single number operation.

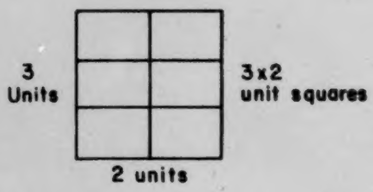


Fig. S

The "Rectangular Region" Model

May we remind you that the idea of multiplying $\frac{2}{3}$ and $\frac{1}{5}$ was not invented for the purpose of writing arithmetic books. Instead, people found some applications in which the numbers $\frac{2}{3}$ and $\frac{1}{5}$ appeared and also $\frac{2}{15}$ appeared. For instance, in Fig. T we see a unit square separated into 15 congruent rectangles. The measure of the shaded region is $\frac{2}{15}$ square units. On the other hand, we have already used the operation of multiplication to compute areas of rectangles having dimensions that are whole numbers. Hence it is natural to say: Let us call $\frac{2}{15}$ the product of $\frac{2}{3}$ and $\frac{1}{5}$ and write $\frac{2}{3} \times \frac{1}{5} = \frac{2}{15}$.

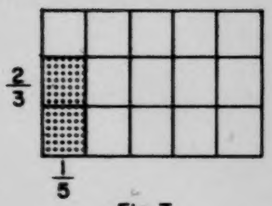


Fig.T

Logically, $\frac{2}{3} \times \frac{1}{5}$ is a meaningless symbol until we define it. It could mean anything we choose. Our choice of $\frac{2}{15}$ for a meaning seems, however, a useful one, and indeed it is.

Yet, children need many more examples before they can see the general rule that in multiplying rational numbers named by fractions we multiply the numerators and multiply the denominators.

We should recognize that although the formal introduction of $\frac{2}{3} \times \frac{1}{5}$ is deferred until the sixth grade the development is anticipated by many earlier experiences. Among them are: the identification of a fraction with a region and the various steps in finding the measure of a region.

The "Number Line" Model

The product of two whole numbers also can be visualized on the number line. A few natural generalizations to products of rational numbers can be made from these kinds of experiences.

For example, if we can think of 3×2 as illustrated by Figure U, (a), then it is natural to identify $3 \times \frac{1}{4}$ with the situation pictured in Figure U, (b).

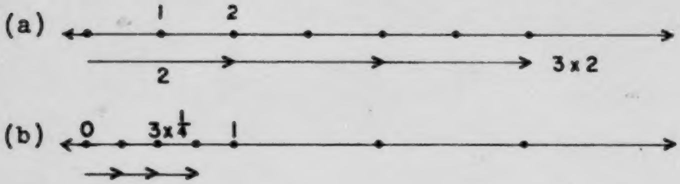


Fig. U

In the same way we can identify, for example, $3 \times \frac{2}{7}$ with

$$\frac{2}{7} + \frac{2}{7} + \frac{2}{7} = \frac{6}{7}.$$

Again, if a man walks 4 miles each hour, then (2×4) miles is the distance he walks in 2 hours. (Fig. V) Once more it is also natural to relate $\frac{1}{2} \times 4$ with a distance he travels--this time with his distance in $\frac{1}{2}$ hour.

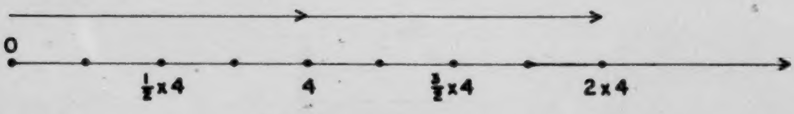


Fig. V

Suppose, now, that a turtle travels $\frac{1}{5}$ mile in an hour. In 2 hours, it travels $2 \times \frac{1}{5}$, or $\frac{2}{5}$ miles. We identify the product $\frac{2}{3} \times \frac{1}{5}$ with the distance it travels in $\frac{2}{3}$ of an hour.

Fig. W diagrams the turtle's travels.

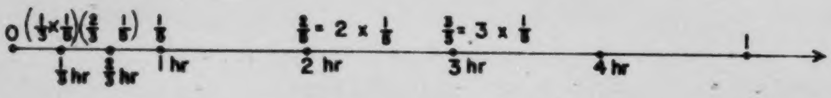

0
$(\frac{1}{3} \times \frac{1}{5})(\frac{2}{3}\ \frac{1}{5})$
$\frac{1}{5}$
$\frac{2}{5} = 2 \times \frac{1}{5}$
$\frac{3}{5} = 3 \times \frac{1}{5}$
1
$\frac{1}{3}$ hr
$\frac{2}{3}$ hr
1 hr
2 hr
3 hr
4 hr

Fig. W

We locate $\frac{2}{3}$ of $\frac{1}{5}$ on the number line by locating $\frac{1}{5}$, cutting the $\frac{1}{5}$ segment into 3 congruent segments, and counting off two of them, as in Figure X.

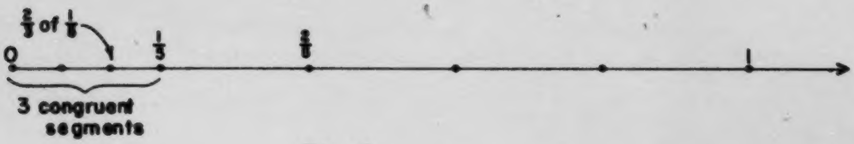

$\frac{2}{3}$ of $\frac{1}{5}$
0
$\frac{1}{5}$
$\frac{2}{5}$
1
3 congruent segments

Fig. X

More specifically, we first cut the unit segment into 5 congruent segments. Then each of these is cut into 3 congruent segments. We thus have 3 × 5 segments. We counted 2 × 1 of them. We see that $\frac{2}{3}$ of $\frac{1}{5}$ is associated with the point $\frac{2 \times 1}{3 \times 5}$.

We had two numbers: $\frac{2}{3}$ and $\frac{1}{5}$. When we talk about $\frac{2}{3}$ of $\frac{1}{5}$ we are explaining a situation in which we have a pair of numbers ($\frac{2}{3}$ and $\frac{1}{5}$) associated with a third ($\frac{2}{15}$). We have, in short, an operation; it is natural to see whether it is an operation we know. We find that it is.

We already had agreed, using the rectangle model, that

$$\frac{2}{3} \times \frac{1}{5} = \frac{2 \times 1}{3 \times 5} = \frac{2}{15}.$$

Hence, we now see that we can identify $\frac{2}{3}$ of $\frac{1}{5}$ with $\frac{2}{3} \times \frac{1}{5}$.

Moreover, we notice that if we use the idea of travel on the number line (Fig. W) it is again natural to identify $\frac{2}{3} \times \frac{1}{5}$ with $\frac{2}{3}$ of $\frac{1}{5}$.

The "Sets of Objects" Model

We speak, in everyday usage, of $\frac{2}{3}$ of a dozen of eggs. We visualize this as the result of separating a finite set of 12 objects into 3 subsets each with the same number of objects and then uniting 2 of the subsets. The relation between this concept and that involving a 12-inch segment can be seen from Figure Y.

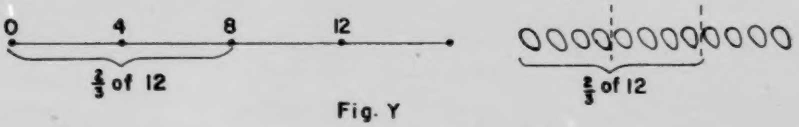


Fig. Y

We sometimes use such examples with very young children to emphasize the idea of $\frac{2}{3}$. But this is a little misleading, for we should note that $\frac{2}{3}$ of 12 is again a situation involving two numbers, $\frac{2}{3}$ and 12. Again we can verify that $\frac{2}{3}$ of 12 is computed by finding $\frac{2}{3} \times 12$.

Summary

From the standpoint of defining the operation of multiplication for rational numbers, it would be entirely sufficient to use one interpretation. However, because products of rational numbers are used in many types of problem situations the child ought to recognize that the definition does fit the needs of each.

$\frac{2}{3} \times \frac{1}{4}$ can be visualized as:

(1) the area in square inches of a rectangle with length $\frac{2}{3}$ in. and width $\frac{1}{4}$ in.

(2) the length of a line segment formed by taking $\frac{2}{3}$ of a $\frac{1}{4}$ inch segment.

Out of the number line model come many problem situations. For example, if a car travels $\frac{1}{4}$ mile per minute, it travels $\frac{2}{3} \times \frac{1}{4}$ miles in $\frac{2}{3}$ minute.

Moreover, $\frac{2}{3} \times 12$ can be interpreted in the ways noted and also can be related to finite sets. We use $\frac{2}{3} \times 12$ where we want to find the number of eggs in $\frac{2}{3}$ of a dozen.

Mathematics is powerful because a single mathematical idea (like $\frac{2}{3} \times \frac{1}{4}$) often has many applications. Children can fully understand a product like $\frac{2}{3} \times \frac{1}{4}$ only when they have had experiences with several applications.

We should observe that our definition of the product of two rational numbers is consistent with what we already know about whole numbers. We know that 2×3, for example, is another name for 6. All is well, for the product of $\frac{2}{1}$ and $\frac{3}{1}$, as computed by our definition, is $\frac{6}{1}$, and $\frac{6}{1}$ names the same number as 6. We note, too, that our definition leads us to $3 \times \frac{1}{4} = \frac{3}{1} \times \frac{1}{4} = \frac{3}{4}$, as was anticipated earlier.

We should notice, too, that although our method for finding the product is expressed in terms of specific names for the factors, the product is not changed if we change the names. For example, $\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$. Renaming $\frac{1}{2}$ and $\frac{3}{4}$ we have

$$\frac{3}{6} \times \frac{6}{8} = \frac{18}{48}.$$

$\frac{18}{48}$ is another name for $\frac{3}{8}$.

Properties of Multiplication of Rational Numbers

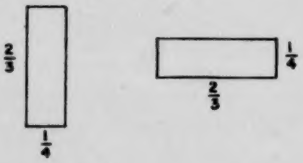
Again our definition has convenient by-products. For, we observe that the associative and commutative properties hold multiplication of rational numbers. They hold here as a direct result of the same properties for multiplication of whole numbers.

The following example shows how we may explain the commutative property of multiplication.

Our rule for multiplication tells us that $\frac{2}{3} \times \frac{1}{4} = \frac{2 \times 1}{3 \times 4}$. Our rule tells us also that $\frac{1}{4} \times \frac{2}{3} = \frac{1 \times 2}{4 \times 3}$. We know, however, that $1 \times 2 = 2 \times 1$ and $3 \times 4 = 4 \times 3$. These facts are instances of the commutative property of multiplication of whole numbers. Hence we see: $\frac{2}{3} \times \frac{1}{4} = \frac{1}{4} \times \frac{2}{3}$.

We also can visualize this property using rectangular regions as in Figure Z.

Figure Z



The associative property of multiplication holds, in essence, because it holds for whole numbers. From the associative property, we can compute $\frac{1}{2}(4 \times 5)$ as either $\frac{1}{2} \times (4 \times 5)$ or $(\frac{1}{2} \times 4) \times 5$--a fact which is sometimes helpful in using the formula for the area of a triangle.

We observe, too, an interesting multiplication property of 0. Our rule for multiplying two rational numbers named by fractions leads directly to the conclusion that the product of 0 and any rational number is 0:

$$0 \times \frac{2}{3} = \frac{0}{1} \times \frac{2}{3} = \frac{0 \times 2}{1 \times 3} = \frac{0}{3} = 0.$$

Of interest, too, is the multiplication property of 1. It is easy to see that the product of 1 and any number is the number:

$$1 \times \frac{2}{3} = \frac{1}{1} \times \frac{2}{3} = \frac{1 \times 2}{1 \times 3} = \frac{2}{3}.$$

Again this is a direct result of the same property of whole numbers. Now 1, of course, has many names. One of them, for example, is $\frac{3}{3}$. When we multiply $\frac{4}{5}$ by 1 we can write

$$1 \times \frac{4}{5} = \frac{3}{3} \times \frac{4}{5} = \frac{3 \times 4}{3 \times 5} = \frac{12}{15} = \frac{4}{5}.$$

This result shows that multiplying the numerator and denominator of a fraction by the same counting number is equivalent to multiplying 1 by the number named by the fraction.

Finally, the distributive property for rational numbers is an outcome of our definition. The distributive property tells us that, for example, $\frac{2}{3}(\frac{1}{5} + \frac{2}{5}) = (\frac{2}{3} \times \frac{1}{5}) + (\frac{2}{3} \times \frac{2}{5})$. Our area picture helps us to understand this easily (Fig. AA) The smaller rectangles have areas $\frac{2}{3} \times \frac{1}{5}$ and $\frac{2}{3} \times \frac{2}{5}$. The area of their union is $\frac{2}{3}(\frac{1}{5} + \frac{2}{5})$, or $\frac{2}{3} \times \frac{3}{5}$.

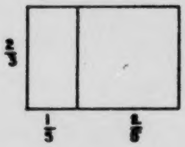


Fig. AA

The distributive property is useful in computing a product like $5 \times 3\frac{1}{2}$. We can say:

$$5 \times 3\frac{1}{2} = (5 \times 3) + (5 \times \frac{1}{2}) = 15 + \frac{5}{2} = 17\frac{1}{2}.$$

We can also recognize that we have essentially applied the distributive property in writing

$$\frac{3}{4} = 3 \times \frac{1}{4} = (1 + 1 + 1) \times \frac{1}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}.$$

Before leaving the topic of multiplication of rational numbers, we ought to notice that the product of certain pairs of rational numbers is 1. For example, the product of $\frac{2}{3}$ and $\frac{3}{2}$ is 1. The number $\frac{2}{3}$ is called the reciprocal of $\frac{3}{2}$, and $\frac{3}{2}$ is the reciprocal of $\frac{2}{3}$. The reciprocal of a number is the number it must be multiplied by to give 1 as a product. Every rational number except 0 has exactly one reciprocal. When the number is named by a fraction, we can easily find the reciprocal by "turning the fraction upside down". Thus the reciprocal of $\frac{3}{8}$ is $\frac{8}{3}$.

In particular, the reciprocal of the whole number 2 is $\frac{1}{2}$, which can be verified easily since $2 \times \frac{1}{2} = 1$.

0 has no reciprocal. For we know that the product of 0 and every rational number is 0. Hence there is no number we can multiply by 0 to give 1.

Division

In the rational number system, as in the counting numbers, we want to use division to answer questions of "what must we multiply?". In a division situation we are given one factor and a product. Thus $\frac{5}{3} \div \frac{3}{4}$ is the number such that: $(\frac{5}{3} \div \frac{3}{4}) \times \frac{3}{4} = \frac{5}{3}$. In order to compute $\frac{5}{3} \div \frac{3}{4}$ we must solve:

$$\frac{3}{4} \times \frac{m}{n} = \frac{5}{3}$$

where m and n are counting numbers.

To acquire an understanding of the division process, children need many concrete experiences in its use. These experiences parallel those with multiplication, since division problems can be interpreted as problems in finding an appropriate multiplier. Thus typical problem situations include: 1) finding the width of a rectangle when the length and area are known;

2) finding what fractional part one set is of another;
3) finding the number of segments of given length that can be made by cutting a given segment.

The Idea of Reciprocal and Division

We already have seen that the product of any number and its reciprocal is 1. For example,

$$\frac{2}{5} \times \frac{5}{2} = \frac{2 \times 5}{5 \times 2} = 1.$$

We also know that $1 \div \frac{2}{5} = n$ means $\frac{2}{5} \times n = 1$. Thus, the reciprocal of $\frac{2}{5}$ (which is $\frac{5}{2}$) is the number by which one can multiply $\frac{2}{5}$ to obtain the product 1.

Now suppose we want to find the number m such that

$$\frac{2}{5} \times m = 3.$$

Since $1 \times 3 = 3$, we can write: $\frac{2}{5} \times m = (1 \times 3)$. But since $\frac{2}{5} \times \frac{5}{2} = 1$, we also can write: $\frac{2}{5} \times m = (\frac{2}{5} \times \frac{5}{2}) \times 3$. Using the associative property, we can write again: $\frac{2}{5} \times m = \frac{2}{5} \times (\frac{5}{2} \times 3.)$ We now see that $m = \frac{5}{2} \times 3$.

We can use the same reasoning to compute $\frac{5}{3} \div \frac{3}{4}$. We must solve:

$$\frac{3}{4} \times m = \frac{5}{3}.$$

We know that $\frac{3}{4} \times \frac{4}{3} = 1$ and that the product of a number and one is the number itself. So,

$$(\frac{3}{4} \times \frac{4}{3}) \times \frac{5}{3} = \frac{5}{3}$$

and

$$\frac{3}{4} \times (\frac{4}{3} \times \frac{5}{3}) = \frac{5}{3}.$$

Therefore, $m = \frac{4}{3} \times \frac{5}{3} = \frac{4 \times 5}{3 \times 3} = \frac{20}{9}$.

We note: To *divide* $\frac{5}{3}$ by $\frac{3}{4}$, we *multiply* $\frac{5}{3}$ by the *reciprocal* of $\frac{3}{4}$.

We have seen a way in which we can derive the general rule: To divide by a non-zero rational number, multiply by its reciprocal.

We see that for whole numbers our rule gives the results we would expect. For example,

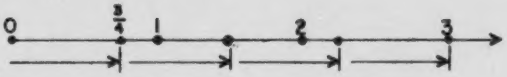
$$6 \div 2 = \frac{6}{1} \div \frac{2}{1} = \frac{6}{1} \times \frac{1}{2} = \frac{6}{2} = 3.$$

In particular, the reciprocal of 1 is 1, since $1 \times 1 = 1$. Thus when we divide a rational number by 1 we multiply it by 1 and obtain the original number, as we would expect.

Our rule for division identifies dividing by a number with multiplying by its reciprocal. Thus when we wish to find $\frac{1}{3}$ of 18 we may use either multiplication by $\frac{1}{3}$ or division by 3.

We now see that: $\frac{3}{4} = 3 \times \frac{1}{4} = 3 \div 4$. One important interpretation of $\frac{3}{4}$ as the result of dividing 3 by 4 can be visualized using line segments. (Fig. BB)

$\frac{3}{4}$ can be seen, too as the answer to the question "How many 4's in 3 ?" More precisely, $\frac{3}{4}$ is the number by which we must multiply 4 to get 3.



4 congruent segments

Fig. BB

Some texts for later grades define rational numbers by using the idea of division. That is, $\frac{3}{4}$ is defined from the outset as the number x satisfying $4x = 3$.

The set of counting numbers is not closed under division--that is, with only counting numbers at our disposal, we can not solve an equation like $4 \times n = 3$. But having introduced the set 7 rational numbers, we can always solve an equation of this type. We can divide any rational number by any number different from 0. Hence the set, made up of all the rational numbers except 0, is closed under division. We can interpret the extension of our idea of number from the counting numbers to the rational numbers as a successful effort to obtain a system of numbers that is closed under division.

It is an interesting paradox that now, having defined division by a rational number as multiplication by the reciprocal of the number, we could really get along without division entirely, since to divide by a number we can always multiply by its reciprocal. Later, we introduce the negative numbers to make a system closed under subtraction. Once we have done so, subtracting a number can be replaced by adding the opposite.

Fractions--A Symbol for a Pair of Rational Numbers

Thus far, we have restricted our use of fraction to that of being a symbol naming a pair of whole numbers. Let us now give meaning to symbols like $\frac{\frac{3}{2}}{\frac{6}{1}}$, $\frac{1.5}{.5}$, etc. in which the pairs of numbers are rational numbers instead of whole numbers.

We call that we already know $\frac{3}{4}$ and $3 \div 4$ are two names for the same number. That is, for $4 \times n = 3$, $n = \frac{1}{4} \times \frac{3}{1} = \frac{3}{4}$. This suggests that we might say that the symbol $\frac{\frac{3}{2}}{6}$ will mean $\frac{3}{2} \div 6$, and the symbol $\frac{1.5}{.5}$ will mean $1.5 \div 5$.

A Definition

When we say, let $\frac{3}{2} \div 6 = \frac{\frac{3}{2}}{6}$, we are defining the meaning of those symbols which hitherto have had no meaning for us. There is nothing illogical with defining a new symbol in any way we like. However, simply assigning to $\frac{3}{2} \div 6$ the symbol $\frac{\frac{3}{2}}{6}$ does not permit us to treat this new symbol immediately as if it were a fraction of the kind with which we are familiar. For example, although we know that $\frac{3}{4} = \frac{2 \times 3}{2 \times 4}$ we cannot be certain that $\frac{\frac{3}{2}}{6} = \frac{2 \times \frac{3}{2}}{2 \times 6}$. Too, just because we know $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$, we cannot conclude (without argument) that $\frac{1}{.1} + \frac{1}{.1} = \frac{2}{.1}$.

In practice, children in the elementary school are unlikely to add or to multiply many numbers named by these new fractions. Yet, examples as $\frac{1.5}{.5}$ will be familiar when using decimal names in division of rational numbers. In later years, they can find solutions to such examples as $12\frac{1}{2}\%$ of 120 by solving

$$\frac{12.5}{100} = \frac{n}{120}.$$

Thus, it seems necessary to "know" if it is possible to multiply the numerator and denominator of $\frac{1.5}{.5}$ by 10 to obtain another name for the same number.

Again, let us make some observations about division. Does multiplying the dividend and divisor by the same number change the result? We observe

$$6 \div 2 = 3$$

$$(6 \times 2) \div (2 \times 2) = 3 \quad \text{or} \quad 12 \div 4 = 3$$

$$(6 \times \tfrac{1}{2}) \div (2 \times \tfrac{1}{2}) = 3 \quad \text{or} \quad \tfrac{6}{2} \div \tfrac{2}{2} = 3 \div 1 = 3.$$

We also need to be sure that when we multiply the numerator and denominator of a fraction, as, $\frac{\frac{5}{3}}{\frac{3}{4}}$, by the same number, we obtain a new fraction equal to the original one.

Does $\frac{5}{3} \div \frac{3}{4} = \frac{\frac{5}{3}}{\frac{3}{4}}$?

We know that $\frac{5}{3} \div \frac{3}{4} = \frac{3}{4} \times n = \frac{5}{3}$ and $n = \frac{4}{3} \times \frac{5}{3} = \frac{20}{9}$.

Let us now multiply both numerator and denominator of the fraction $\frac{\frac{5}{3}}{\frac{3}{4}}$ by the same number $\frac{4}{3}$.

$$\frac{\frac{5}{3} \times \frac{4}{3}}{\frac{3}{4} \times \frac{4}{3}} = \frac{\frac{5 \times 4}{3 \times 3}}{\frac{3 \times 4}{4 \times 3}}$$

But $\frac{3 \times 4}{4 \times 3} = \frac{12}{12} = 1$ and $\frac{5 \times 4}{3 \times 3} = \frac{20}{9}$.

Is $\frac{\frac{20}{9}}{1}$ the same as $\frac{20}{9}$?

If it is, then $\frac{20}{9} \div 1 = \frac{20}{9}$.

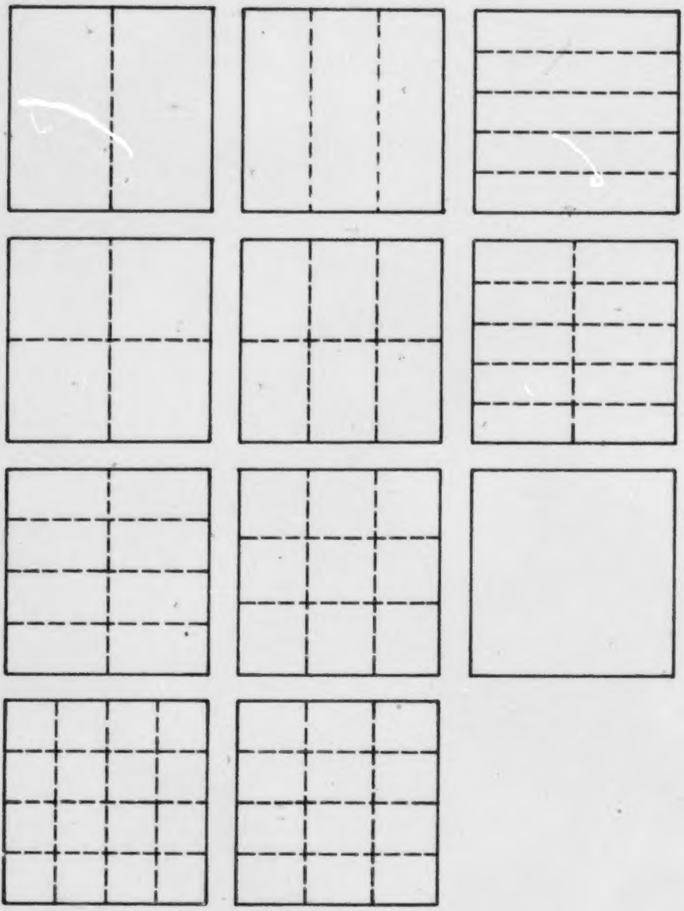
We do know that this is true since the product of 1 and a rational number is that same rational number.

Thus $\frac{5}{3} \div \frac{3}{4} = \frac{20}{9}$ and $\frac{\frac{5}{3}}{\frac{3}{4}} = \frac{20}{9}$; so, $\frac{5}{3} \div \frac{3}{4} = \frac{\frac{5}{3}}{\frac{3}{4}}$.

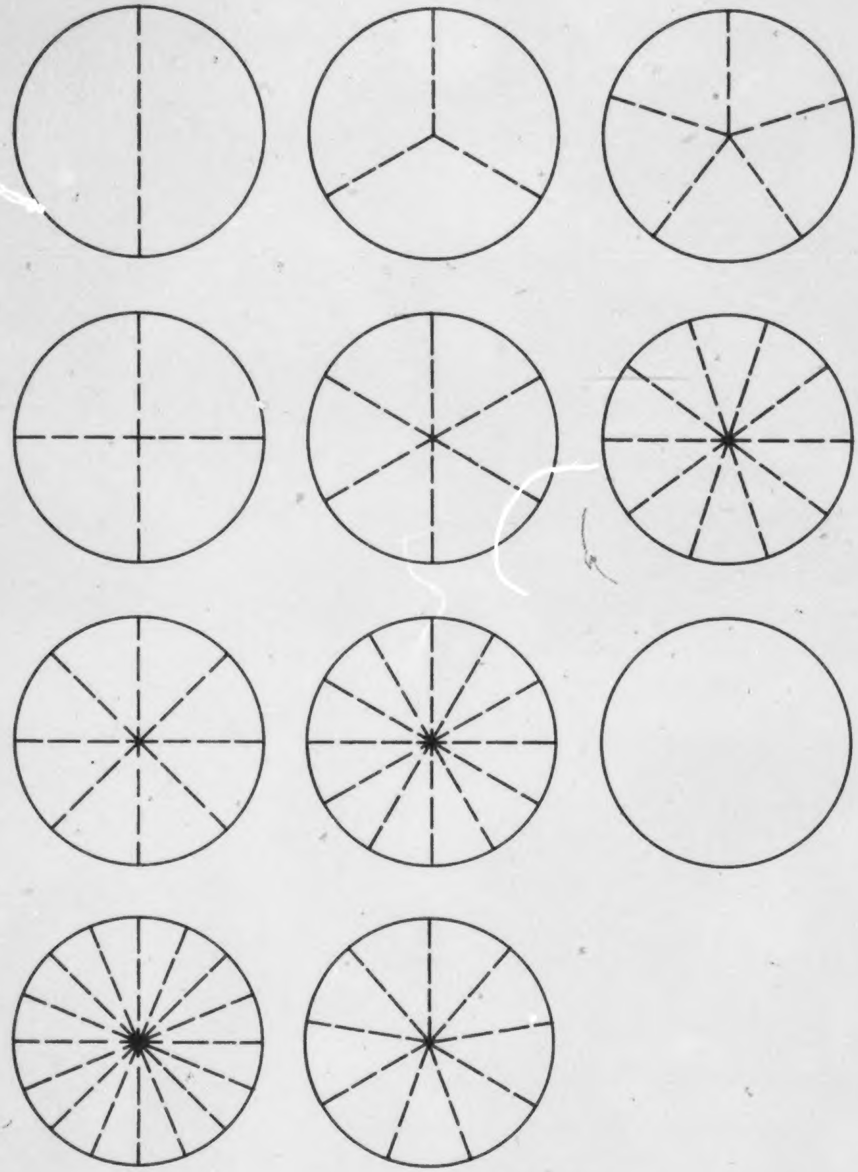
MATERIALS

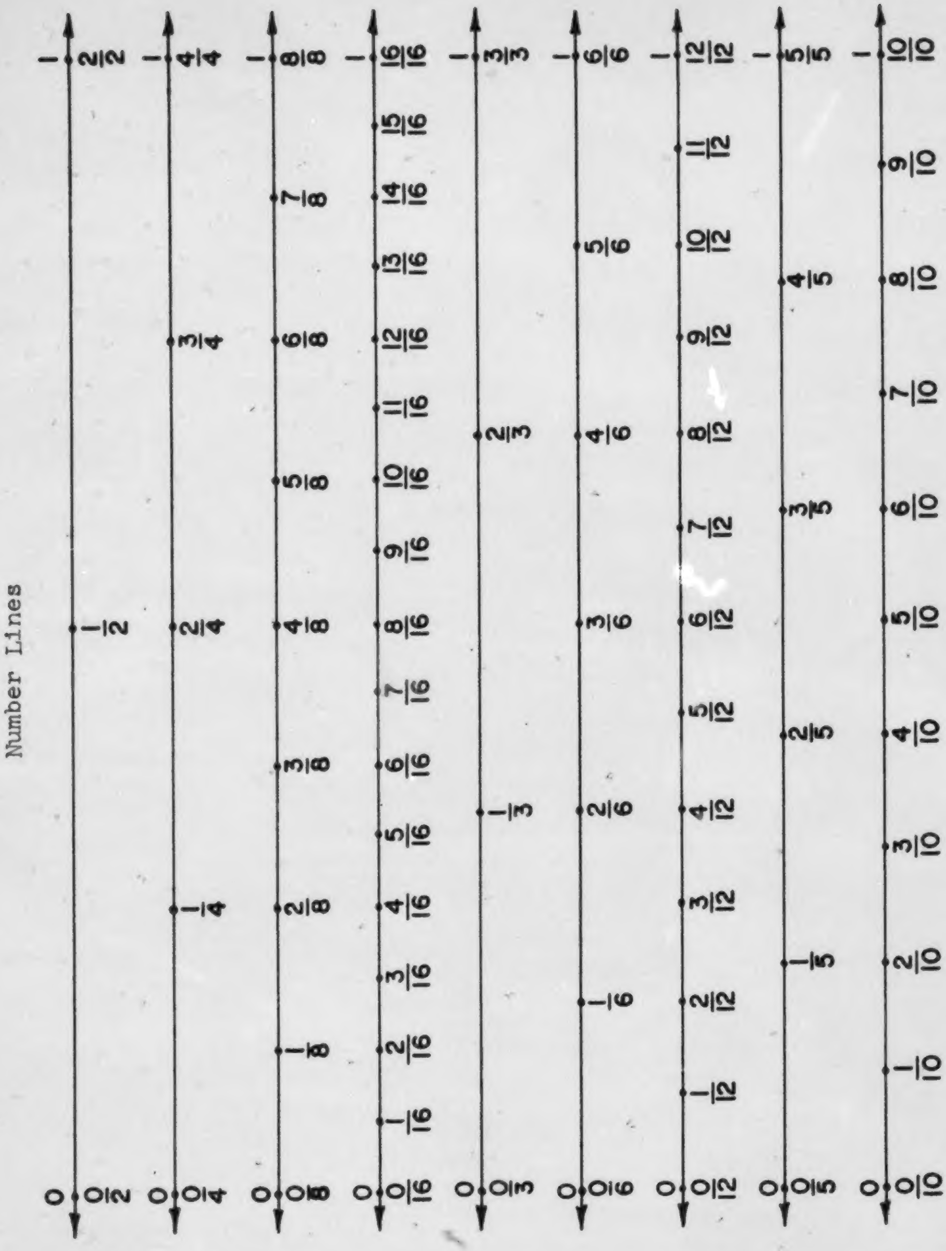
It is important that extensive use be made of materials in developing understanding of the rational numbers and of the operations of addition and subtraction of rational numbers. Some materials which have been found useful are suggested on the next few pages. These may be supplemented by other available materials.

Teachers will find copies of these cards made on foot square cardboard useful throughout the chapter. Colored acetate may be used to indicate shaded areas.



Models of circular regions can be copied on cardboard or undecorated paper plates.



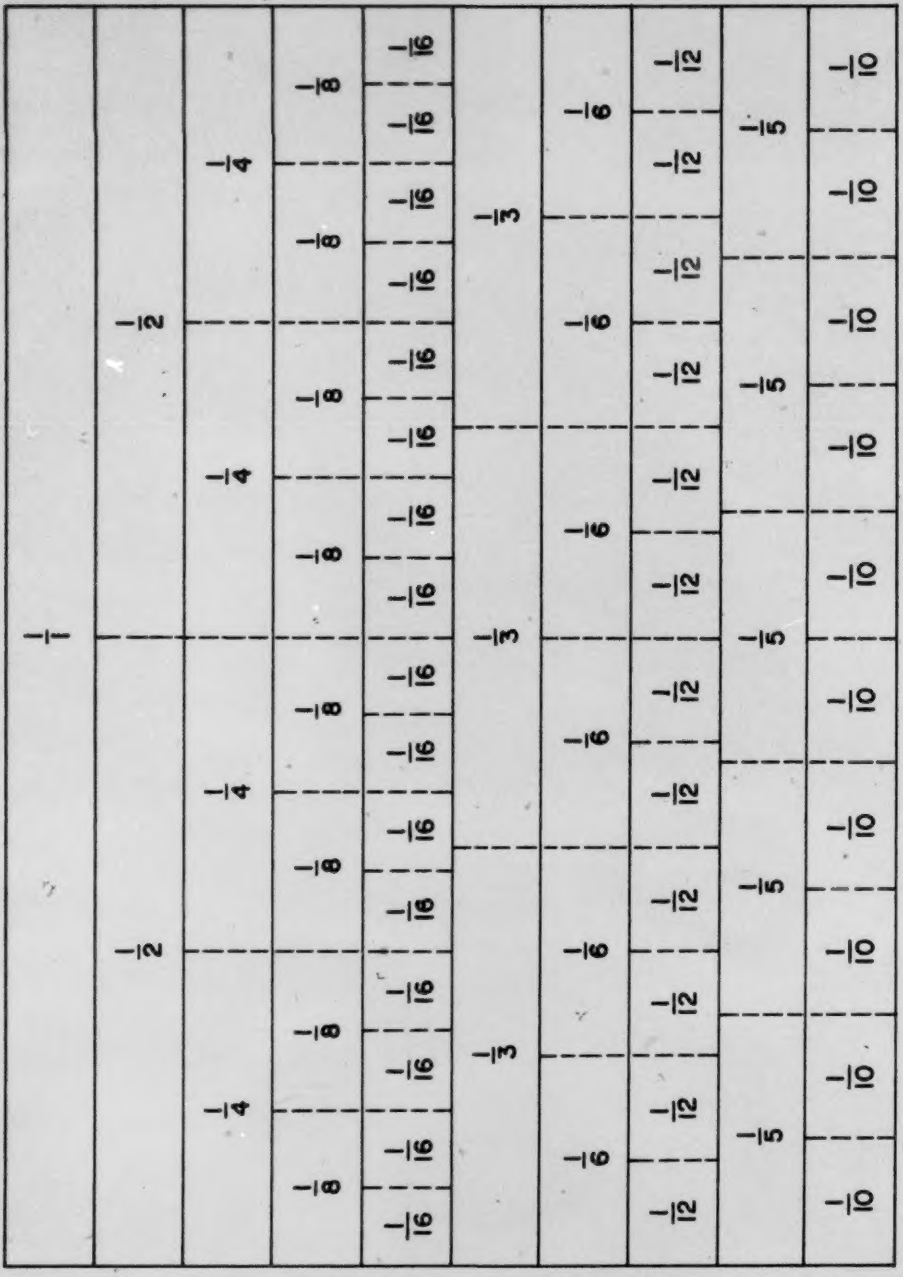


Number Lines

0 0/2 1/2 1 2/2
0 0/4 1/4 2/4 3/4 1 4/4
0 0/8 1/8 2/8 3/8 4/8 5/8 6/8 7/8 1 8/8
0 0/16 1/16 2/16 3/16 4/16 5/16 6/16 7/16 8/16 9/16 10/16 11/16 12/16 13/16 14/16 15/16 1 16/16
0 0/3 1/3 2/3 1 3/3
0 0/6 1/6 2/6 3/6 4/6 5/6 1 6/6
0 0/12 1/12 2/12 3/12 4/12 5/12 6/12 7/12 8/12 9/12 10/12 11/12 1 12/12
0 0/5 1/5 2/5 3/5 4/5 1 5/5
0 0/10 1/10 2/10 3/10 4/10 5/10 6/10 7/10 8/10 9/10 1 10/10

This page of number lines might be enlarged for class use. Dittoed copies of this page might be prepared by teacher and distributed to pupils.

Fraction Chart



1/1
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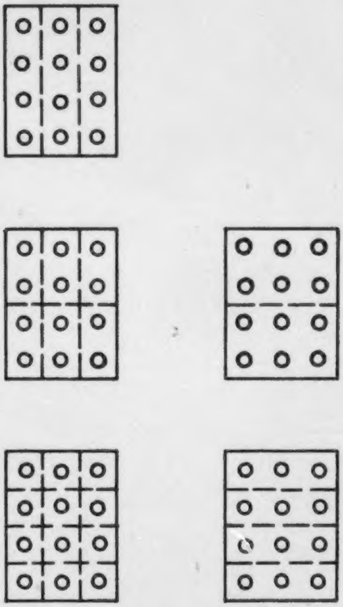
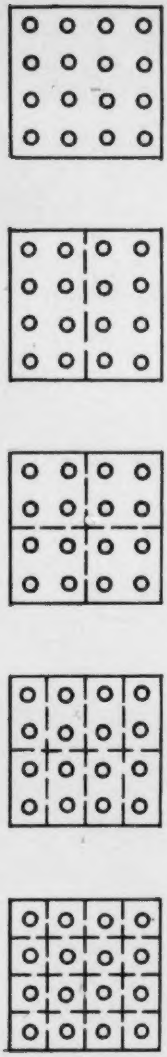
1/10
1/10

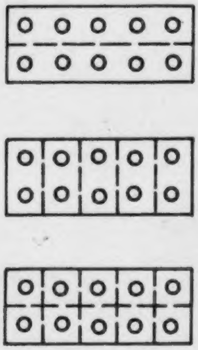
This model of a fraction chart might be enlarged for class use. Dittoed copies could also be prepared by teachers for each child.



SET B



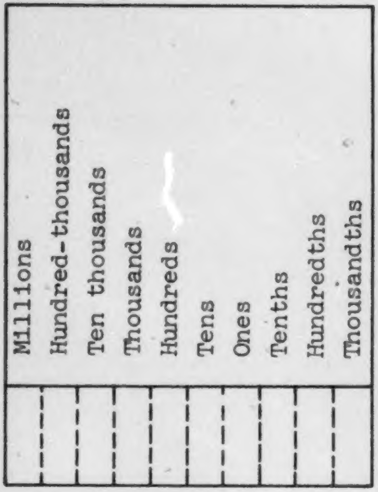
SET A

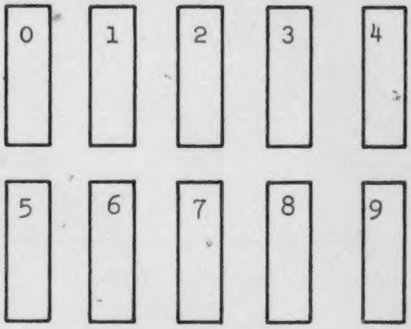


SET C

These arrays may be used to develop the concept of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc. of a set of objects. Colored acetate may be used with them to indicate $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, etc. of a set of objects.

Pocket Chart and Cards for Decimals


Millions
Hundred-thousands
Ten thousands
Thousands
Hundreds
Tens
Ones
Tenths
Hundredths
Thousandths


0
1
2
3
4
5
6
7
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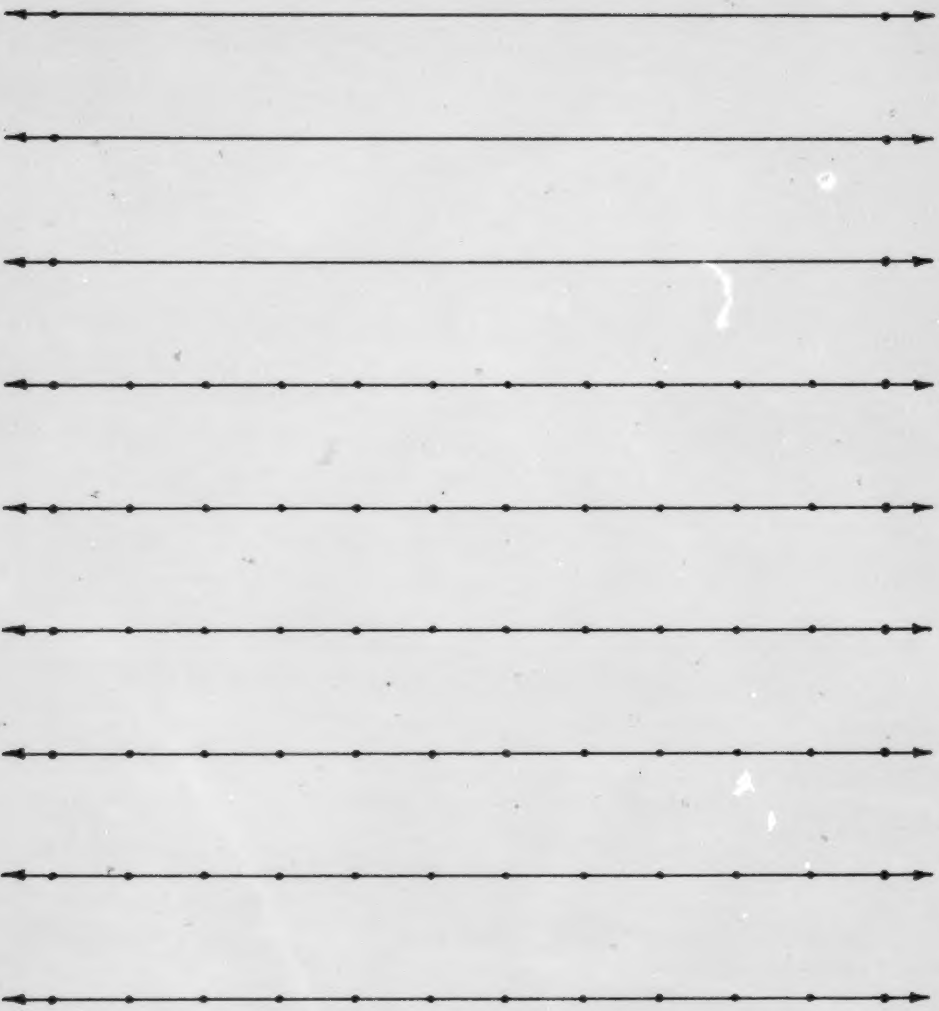
9

Above is a model of a pocket chart and cards to accompany it for practice as needed in place value work with decimals. Several duplicates of each card should be made.

A sheet of number lines similar to this can be dittoed for children to use throughout unit.



TEACHING THE UNIT

MEANING OF RATIONAL NUMBERS

Objective: To review

a. the meaning of rational numbers.

b. correct use of the terms symbol, fraction.

c. meaning of numerator and denominator.

d. the use of a rational number to describe the measure of a region, a segment, or a subset of a set of objects.

Materials: Flannel board with models of circular and square regions separated into congruent parts.
Number lines.
Arrays or sets of objects.

Vocabulary: Rational number, fraction, numerator, denominator, whole number, separate, measure, congruent, circular, unit region, region, line segment, unit segment, represent, union, set.

Suggested Teaching Procedure

Have children read together and discuss "Meaning of Rational Numbers" in their books. Before having children do Exercise Set 1, teacher shows work with concrete materials such as a flannelboard and models of congruent regions, number lines, and sets of objects or arrays.

Models of congruent regions and sets shown under "Materials" can be used throughout this development. Colored acetate is very useful to indicate shaded regions.

Begin by shading $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc. of model regions and arrays. Ask questions like the following:

(a) Into how many congruent regions is the unit region separated?

(b) How many regions are shaded?

(c) What rational number best describes its measure? Record this measure.

(d) What does the denominator tell you?

(e) What does the numerator tell you?

Ask similar questions about the unshaded regions.

Continue with models shading $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, etc. and asking similar questions.

It is important to emphasize that:

(a) measure is a comparison with the unit.

(b) regions whose measures are rational numbers have meaning only when the unit region is specified.

Chapter 6

ADDITION AND SUBTRACTION OF RATIONAL NUMBERS

MEANING OF RATIONAL NUMBERS

Because of his way of life, early man needed only whole numbers. We can think of reasons why he came to need other numbers as time went by. For example, he might have wanted to trade more than 2 but less than 3 hides for a weapon. He might have wished to tell someone that there was some food but not enough for a meal. He could not have handled these situations with whole numbers alone.

Today you would have great difficulty in making yourself understood if you could use only whole numbers. Suppose you knew only whole numbers. Could you describe any of these with a whole number?

(a) A trip that took less than one day

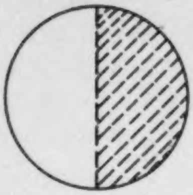
(b) The amount of candy you get when you share a candy bar with two friends

(c) The number of books you read this summer, if you read more than 8 and less than 9

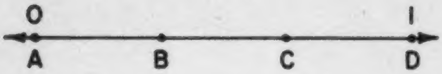
You would have even more difficulty in mathematics. If you could use only whole numbers, there would be no result for such operations as $2 \div 5$ or $8 \div 3$. Another set of numbers helps you find answers to such operations. This set of numbers is called the set of rational numbers.

Rational numbers are often used to describe the measure of a region, segment, or set of objects.

Exploration



One-half of the circular region is shaded. The numeral for one-half is $\frac{1}{2}$.



Points B and C separate $\overline{AD}$ into 3 congruent segments. If the measure of $\overline{AD}$ is 1, the measure of $\overline{AC}$ is two-thirds. The numeral for two-thirds is $\frac{2}{3}$.

Set A = {Tom, Jane, Bill, Ann, Sally}

Three-fifths is the number that best describes what part of Set A the three girls are. The numeral for three-fifths is $\frac{3}{5}$.

One kind of symbol used to name a rational number is called a fraction. A fraction is written with two names for whole numbers separated by a bar. $\frac{2}{3}$, $\frac{3}{5}$ and $\frac{1}{2}$ are fractions.

$$\frac{\text{numeral}}{\text{numeral}}$$

The number named below the bar is called the denominator of the fraction and shows into how many parts of equal measure the unit region, unit segment, or set is separated. The number named above the bar, or numerator, counts the number of these parts that are being used.

1. Look at the circular region above.
   a. Into how many congruent regions is it separated? (2)
   b. Will this number be represented by a numeral written above or below the bar of a fraction? (below)
   c. What is this numeral called? (denominator)

2. a. How many congruent parts of the region are shaded? (1)

   b. Where will you write the numeral that shows this? (above bar)

   c. What is it called? (numerator)

3. Look at the picture of $\overline{AC}$. The measure of $\overline{AC}$ is written $\frac{2}{3}$.

   a. What does the denominator tell you? ($\overline{AD}$ is separated into 3 congruent segments)

   b. What does the numerator tell you? ($\overline{AC}$ represents the union of 2 of these segments)

4. $\frac{3}{5}$ of the members of Set A are girls. What is the relation of the 3 and 5 of the fraction to Set A? (5 represents members in set. 3 represents members who are girls)

Summary

1. A rational number is sometimes used to describe the measure of a region, line segment, or set of objects.

2. Fractions are one of the symbols used to name rational numbers.

3. Fractions are written with 2 names for whole numbers separated by a bar. (The denominator can not be 0.)

$$\frac{4}{5} \quad \frac{\text{numerator}}{\text{denominator}}$$

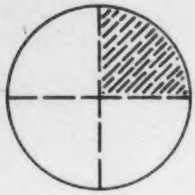
4. The denominator is the number which tells the number of congruent parts into which the unit segment, unit region, or set has been separated.

5. The numerator is the number which counts the number of these congruent parts that are being used.

6. The set of rational numbers includes numbers renamed by numerals like these: $1\frac{3}{8}$, 5, $\frac{2}{2}$, $\frac{4}{3}$, 0, 7.2 and $\frac{1}{2}$.

Exercise Set 1

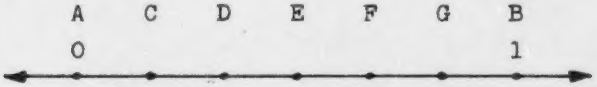
1.



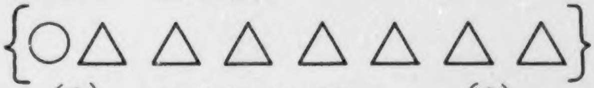
A

The circular region A has been separated into (4) congruent regions. (1) region is shaded. The shaded region is $(\frac{1}{4})$ of the circular region.

2.

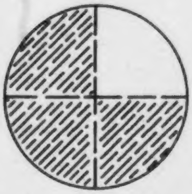


The measure of $\overline{AB}$ is 1. Points C, D, E, F, and G separate $\overline{AB}$ into (6) congruent segments. $\overline{AG}$ is the union of (5) of these congruent segments. The measure of $\overline{AG}$ is $(\frac{5}{6})$.

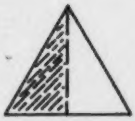
3. Set A = 

There are (8) members in Set A. (7) members are triangles. What rational number describes what part of the members of Set A are triangles? $(\frac{7}{8})$

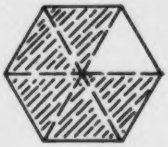
4.



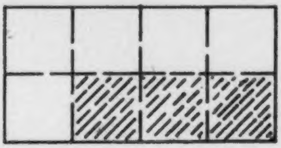
A



B



C

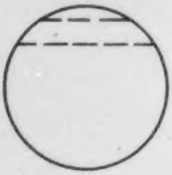


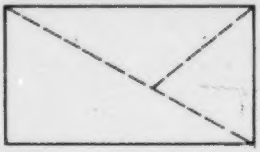
D

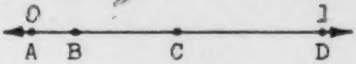
Each unit region above is separated into a number of smaller congruent regions. What rational number best describes the measure of the shaded area of each? The unshaded area?

(A shaded $\frac{3}{4}$, unshaded $\frac{1}{4}$; C shaded $\frac{5}{6}$, unshaded $\frac{1}{6}$
B shaded $\frac{1}{2}$, unshaded $\frac{1}{2}$; 498 D shaded $\frac{3}{8}$, unshaded $\frac{5}{8}$)

5. Which figures below are not separated into thirds? Explain your answers. (None; because regions, segments, subsets are not same size)

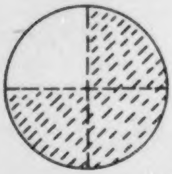




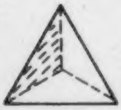


Set A = {▨ ▨ ▨ □ □}

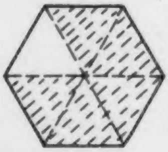
6. Trace the figures below. Then shade a part described by the fraction written below each of the figures.



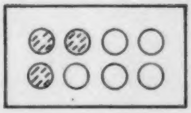
a. $\frac{3}{4}$



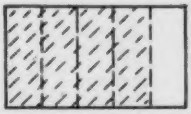
b. $\frac{1}{3}$



c. $\frac{5}{6}$



d. $\frac{3}{8}$



e. $\frac{4}{5}$

(Shade a 3 parts, b 1 part, c 5 parts, d 3 parts, e 4 parts)

7. Draw simple figures and shade parts to show:

a. $\frac{1}{6}$ b. $\frac{5}{8}$ c. $\frac{2}{5}$ d. $\frac{7}{10}$

(Figures will vary)

8. Complete:

Fractions may be used to name (rational) numbers.

Fractions are written with (2) numerals separated by a bar.

The numeral (below) the bar names the (denominator) and tells into how many parts of the same size the unit is separated.

The numeral (above) the bar names the (numerator) and counts the number of parts of the same size being used.

9. Match each rational number named in Column A with its symbol in Column B.

	Column A			Column B
a.	two-fifths	(i.)	g.	$\frac{7}{9}$
b.	seven-ninths	(g.)	h.	$\frac{7}{4}$
c.	four-sevenths	(l.)	i.	$\frac{2}{5}$
d.	five-halves	(k.)	j.	$\frac{9}{7}$
e.	nine-sevenths	(j.)	k.	$\frac{5}{2}$
f.	seven-fourths	(h.)	l.	$\frac{4}{7}$

10. Set A = {set of rational numbers}

Write names for five members of Set A. (Answers will vary)

11. Complete:

Set A = {○○○○}
Set B = {○○○}
Set C = {○○}
Set D = {○}

If the measure of Set D is 1 the measure of Set A is (4). The measure of Set B is (3). The measure of Set C is (2).

12. Use Sets A, B, C, and D in exercise 11 to complete the following:

If the measure of Set A is 1, the measure of Set B is $(\frac{3}{4})$. The measure of Set C is $(\frac{2}{4})$. The measure of Set D is $(\frac{1}{4})$.

RATIONAL NUMBERS ON THE NUMBER LINE

Objectives: To review

a. the use of rational numbers as measures of segments on the number line.

b. the idea that some rational numbers are whole numbers.

c. order of rational numbers on the number line.

d. comparison of rational numbers.

Materials: Number lines, ruler, yardstick

Vocabulary: Label, point, unit segment, simplest name, order, compare, generalization

Suggested Teaching Procedure

Have children work through Exploration (first two pages). Then working together teacher and pupil should build number lines scaled in fourths and eighths, halves, thirds, and sixths, etc.

Have children finish Exploration together. Before having children do the Exercise Set 2, the teacher may:

(a) have children count by rational numbers using number lines, ruler, or yardstick;

(b) examine the order of rational numbers on the number line and their relation to whole numbers;

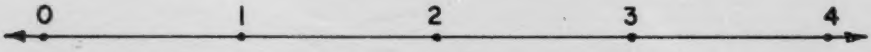
(c) compare rational numbers.

All this can be done on number lines on board, enlarged on paper, or dittoed for children.

RATIONAL NUMBERS ON THE NUMBER LINE

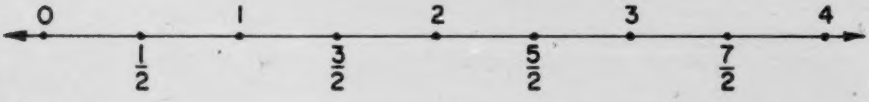
Exploration

1. Draw a number line on your paper. Choose a point and label it 0. Label other points equally spaced in order 1, 2, 3, and 4. Your number line should look like this.

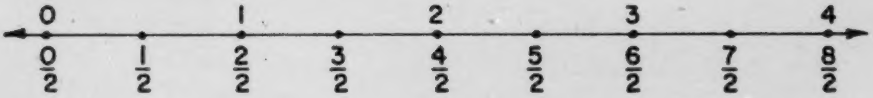


2. Separate each unit segment into two congruent segments. Below the number line, label these points in order $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, and $\frac{7}{2}$.

   Does your number line look like the one below?



3. Which points can now be labeled $\frac{0}{2}$, $\frac{2}{2}$, $\frac{4}{2}$, $\frac{6}{2}$, and $\frac{8}{2}$? Label these points. (0, 1, 2, 3, and 4.)



   Your number line now shows a set of segments, each $\frac{1}{2}$ the length of the original unit segment. The endpoints are labeled with fractions.

4. Look at the fraction labels. What does each denominator tell you? (Each unit segment has been separated into 2 congruent segments. The numerator tells you how many segments of this length have been laid off beginning at 0) What does each numerator tell you? The points you labeled $\frac{0}{2}$, $\frac{2}{2}$, $\frac{4}{2}$, $\frac{6}{2}$, and $\frac{8}{2}$ were already labeled with whole numbers.

5. What other names for points labeled $\frac{0}{2}$, $\frac{2}{2}$, $\frac{4}{2}$, $\frac{6}{2}$, and $\frac{8}{2}$ are shown on the number line? (0, 1, 2, 3, 4) Can a point on the number line have more than one name? (yes)

6. What points could also be labeled $1\frac{1}{2}$, $2\frac{1}{2}$, and $3\frac{1}{2}$? ($\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$) (These numerals are read, "1 and 1 half, 2 and 1 half" etc.)

7. Do $\frac{3}{2}$ and $1\frac{1}{2}$ name the same point? (yes) If so, they are names for the same number. (yes)

8. Do $\frac{5}{2}$ and $2\frac{1}{2}$ name the same point on the number line? (yes)

Summary:

1. A number line may show more than one set of division points.

2. A unit segment on the number line may be separated into any number of congruent segments.

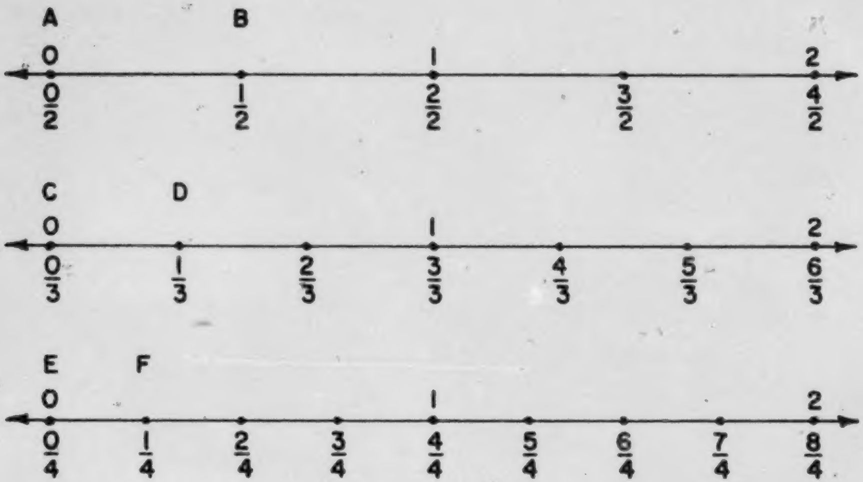
3. The measure of each smaller, congruent segment is a rational number.

4. Some points on the number line can be named by numerals for whole numbers and also by fractions.

5. Numerals like $1\frac{1}{2}$, $2\frac{1}{2}$, and $3\frac{1}{2}$ are read, "1 and 1 half," "2 and 1 half," and "3 and 1 half."

6. Numerals like $1\frac{1}{2}$, $2\frac{1}{2}$, and $3\frac{1}{2}$ can be used as names for certain points on the number line.

Exploration


A
B
0
1
2
$\frac{0}{2}$
$\frac{1}{2}$
$\frac{2}{2}$
$\frac{3}{2}$
$\frac{4}{2}$
C
D
0
1
2
$\frac{0}{3}$
$\frac{1}{3}$
$\frac{2}{3}$
$\frac{3}{3}$
$\frac{4}{3}$
$\frac{5}{3}$
$\frac{6}{3}$
E
F
0
1
2
$\frac{0}{4}$
$\frac{1}{4}$
$\frac{2}{4}$
$\frac{3}{4}$
$\frac{4}{4}$
$\frac{5}{4}$
$\frac{6}{4}$
$\frac{7}{4}$
$\frac{8}{4}$

1. Use the number lines above to answer the following.

   a. What is the measure of $\overline{AB}$? Why? ($\frac{1}{2}$ unit segment is separated into 2 congruent segments. 1 segment is being considered)

   b. What is the measure of $\overline{CD}$? Why? ($\frac{1}{3}$ unit segment has been separated into 3 congruent segments. 1 segment is being considered)

   c. What is the measure of $\overline{EF}$? Why? ($\frac{1}{4}$ unit segment separated into 4 congruent segments. 1 is being considered)

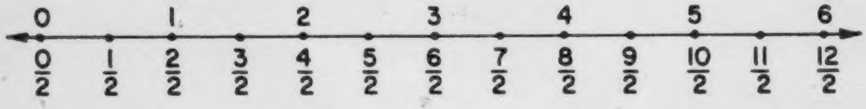
   d. Which of the three rational numbers is the greatest? ($\frac{1}{2}$)

   e. Which of the three rational numbers is the least? ($\frac{1}{4}$)

   f. Arrange the measures of $\overline{AB}$, $\overline{CD}$ and $\overline{EF}$ in order from greatest to least. ($\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$)

   g. What generalization can you make about the order of numbers named by fractions whose numerators are 1?
   (When fractions have numerators of 1, consider the denominators to compare the numbers. The greater the denominator, the smaller the number.)

2.



Use the number line above to answer the following questions:

a. Which rational number is greater, $\frac{3}{2}$ or $\frac{5}{2}$? $(\frac{5}{2})$

b. How can you tell which rational number is greater?
(When fractions have the same denominator, consider the numerators to compare the numbers. The greater the numerator, the larger the number.)

3. Which of the rational numbers in each pair is greater?

a. $\frac{2}{2}$ or $\frac{5}{2}$ $(\frac{5}{2})$

b. $\frac{6}{2}$ or $\frac{8}{2}$ $(\frac{8}{2})$

c. $\frac{7}{2}$ or $\frac{3}{2}$ $(\frac{7}{2})$

4. What are other names for $\frac{0}{2}^{(0)}$, $\frac{4}{2}^{(2)}$ and $\frac{6}{2}^{(3)}$? What are 2 numerals that name:

a. 1 one and 1 half? $(1\frac{1}{2})$ $(\frac{3}{2})$

b. 3 ones and 1 half? $(3\frac{1}{2})$ $(\frac{7}{2})$

c. 5 ones and 1 half? $(5\frac{1}{2})$ $(\frac{11}{2})$

Summary:

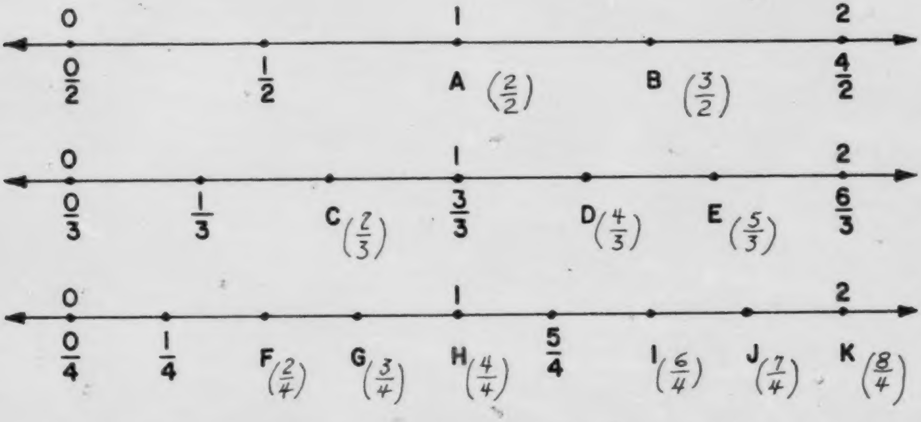
1. To compare rational numbers named by fractions whose numerators are 1, look at the denominators. The greater the number represented by the denominator the smaller the rational number.

2. On the number line, any fraction to the right of another names the greater rational number. Any fraction to the left of another represents the smaller rational number.

3. The order of rational numbers on the number line is the same as the order of whole numbers. As you move to the right along the number line, the rational numbers become greater. As you move to the left, they become smaller.

Exercise Set 2

1. Label the points A-K shown on the number lines below with fractions for rational numbers.


0
1
2
0/2
1/2
A (2/2)
B (3/2)
4/2
0
1
2
0/3
1/3
C (2/3)
3/3
D (4/3)
E (5/3)
6/3
0
1
2
0/4
1/4
F (2/4)
G (3/4)
H (4/4)
5/4
I (6/4)
J (7/4)
K (8/4)

2. Complete each mathematical sentence below. Use <, >, and =. The number lines in exercise 1 will help you.

a. $\frac{2}{3}$ (<) $\frac{3}{4}$

b. $\frac{4}{4}$ (=) $\frac{3}{3}$

c. $\frac{6}{4}$ (>) $\frac{4}{3}$

d. $\frac{4}{2}$ (=) $\frac{6}{3}$

e. $\frac{7}{4}$ (<) $\frac{6}{3}$

f. $\frac{2}{3}$ (>) $\frac{2}{4}$

g. $\frac{1}{4}$ (<) $\frac{1}{3}$

h. $\frac{6}{4}$ (=) $\frac{3}{2}$

3. Which is greater? How can you tell?

a. 4 or 1 (4) e. $\frac{1}{2}$ or $\frac{2}{2}$ $(\frac{2}{2})$

b. $\frac{7}{2}$ or $\frac{5}{2}$ $(\frac{7}{2})$ f. $1\frac{1}{2}$ or $\frac{4}{3}$ $(1\frac{1}{2})$

c. $\frac{4}{2}$ or 1 $(\frac{4}{2})$ g. $1\frac{1}{3}$ or 2 (2)

d. 0 or 2 (2) h. $\frac{5}{4}$ or 1 $(\frac{5}{4})$

(In comparing 2 numbers on the number line, the one to the right is greater.)

4. Arrange in order from least to greatest:

$\frac{7}{4}$, 0, $1\frac{1}{2}$, and $\frac{4}{3}$ $(0, \frac{4}{3}, 1\frac{1}{2}, \frac{7}{4})$

5. Write the whole numbers that are between $\frac{1}{2}$ and $\frac{7}{2}$. (1, 2, 3)

6. $1\frac{1}{2}$, $\frac{3}{2}$, and 1 one and 1 half are all names for the same point on the number line. Write 2 other names for:

a. 1 one and 1 fourth $(1\frac{1}{4}, \frac{5}{4})$

b. $\frac{4}{3}$ (one and one-third, $1\frac{1}{3}$)

7. Write the rational numbers in each set in order of size from least to greatest.

Set A = $\{\frac{1}{4}, \frac{1}{2}, \frac{1}{3}\}$ $(\frac{1}{4}, \frac{1}{3}, \frac{1}{2})$ Set C = $\{\frac{1}{6}, \frac{1}{3}, \frac{1}{12}\}$ $(\frac{1}{12}, \frac{1}{6}, \frac{1}{3})$

Set B = $\{\frac{3}{4}, \frac{3}{5}, \frac{3}{8}\}$ $(\frac{3}{8}, \frac{3}{5}, \frac{3}{4})$ Set D = $\{\frac{5}{10}, \frac{5}{2}, \frac{5}{4}\}$ $(\frac{5}{10}, \frac{5}{4}, \frac{5}{2})$

8. Count by fourths from 0 to 3. Write your answers in a set. If you need help the number line will help you. The set has been started for you. A = $\{\frac{0}{4}, \frac{1}{4},$ $(\frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \frac{5}{4}, \frac{6}{4}, \frac{7}{4}, \frac{8}{4}, \frac{9}{4}, \frac{10}{4}, \frac{11}{4}, \frac{12}{4})$

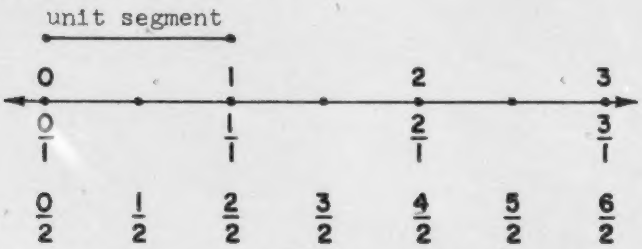
PICTURING RATIONAL NUMBERS ON THE NUMBER LINE

Objective: To help pupils use rational numbers to name different points on the number line.

Vocabulary: Lay off

Materials: Number line on board

Suggested Teaching Procedure

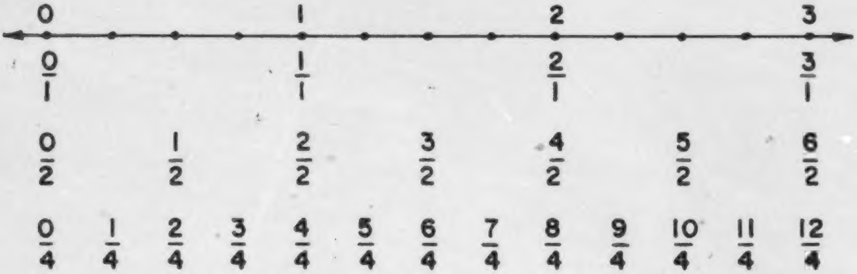


Let us think about the number line pictured above. Compared to the unit segment, what number tells the measure of the segment with end points 0 and $\frac{1}{2}$? What do the 1 and 2 in $\frac{1}{2}$ tell us? (The 2 tells us we are to think of the unit segment as separated into two congruent segments. The 1 tells us that if we start at 0 and lay off one of these congruent segments toward the right, the other endpoint is at $\frac{1}{2}$.) Compared to the unit segment, what number tells the measure of the segment with endpoints 0 and $\frac{3}{2}$? ($\frac{3}{2}$) What do the 3 and 2 in $\frac{3}{2}$ tell us? (The 2 tells us we are to think of the unit segment as separated into two congruent segments. The 3 tells us that if we start at 0 and lay off three of these congruent segments end to end toward the right, the other endpoint is at $\frac{3}{2}$.)

> To emphasize these ideas by repetition, discuss the fractions $\frac{5}{2}$, $\frac{7}{2}$, and so on, using similar questions.

What do the 1 and 3 in the fraction $\frac{3}{1}$ tell us? (The 1 tells us that we are to think of the segment of measure 1. The 3 tells that if we start at 0 and lay off this segment 3 times, end-to-end to the right, the endpoint of the last segment will be on the point $\frac{3}{1}$.)

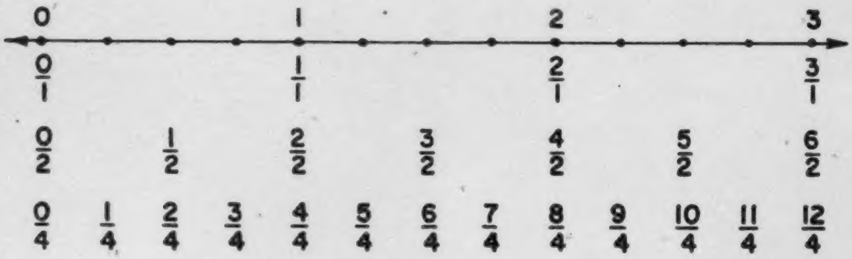
Next, separating the unit segment into 4 (instead of 1 or 2) congruent segments, we locate additional points as before. We label the (old and new) points $\frac{0}{4}$, $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, and so on, as shown below.



Continue, separating unit into 8 congruent segments. The procedure may be continued on other number lines with the unit segments separated into 3, 6, and 12, and 2, 5, and 10 congruent segments.

PICTURING RATIONAL NUMBERS ON THE NUMBER LINE

Exercise Set 3



1. In picturing $\frac{5}{4}$ on the number line, into how many congruent segments do you separate a segment of length 1? (4) Starting at 0, how many times do you lay off to the right a segment of this length to arrive at the point $\frac{5}{4}$? (5)

2. In picturing $\frac{3}{4}$ on the number line, into how many segments of the same length do you separate a segment of length 1? (4) Starting at 0, how many times do you lay off to the right a segment of this length to arrive at the point $\frac{3}{4}$? (3)

3. In picturing 3 on the number line, into how many congruent segments do you separate a segment of length 1 ? (1) Starting at 0, how many times do you lay off to the right a segment of this length to arrive at the point 3? (3)

4. In picturing $\frac{11}{4}$ on the number line, into how many congruent segments do you separate a segment of length 1? (4) Starting at zero, how many times do you lay off to the right a segment of this length to arrive at the point $\frac{11}{4}$? (11)

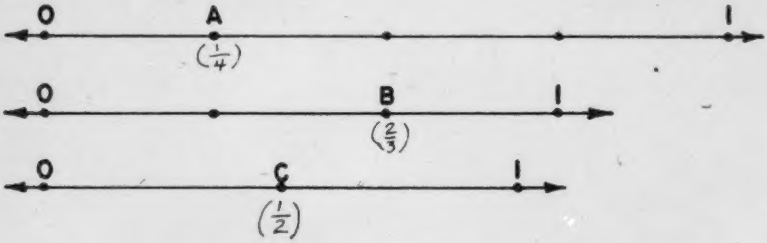
5. In picturing $\frac{6}{2}$ on the number line, into how many congruent segments do you separate a segment of length 1? (2) Starting at 0, how many times do you lay off to the right a segment of this length to arrive at the point $\frac{6}{2}$? (6)

6. How many names are shown on the number line in exercise 1 for

a. 0 $(0, \frac{0}{1}, \frac{0}{2}, \frac{0}{4})$ c. 3 $(3, \frac{3}{1}, \frac{6}{2}, \frac{12}{4})$

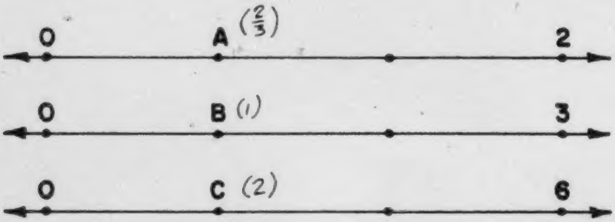
b. $\frac{1}{2}$ $(\frac{1}{2}, \frac{2}{4})$ d. $2\frac{1}{2}$ $(\frac{5}{2}, \frac{10}{4})$

7. Label with fractions points A, B, and C on the number lines below.



BRAINTWISTER

Label with rational numbers the points A, B, and C on the number lines below.



PICTURING RATIONAL NUMBERS WITH REGIONS

Objective: To use rectangular or circular regions to help pupils learn how to write different names for the same rational numbers.

Materials: Number lines, flannel board, and model regions listed under "Materials."

Vocabulary: Rectangle, rectangular region, trace

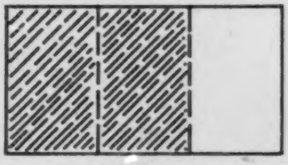
Suggested Teaching Procedure

> Work through Exploration with class. Before having children do Exercise Set 4, teacher will want to work with concrete materials such as models of rectangular regions, models of circular regions, number lines, flannel board, etc., to reinforce learning. She might also wish to begin a chart on which to record and display the many different names for rational numbers like $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, etc.

PICTURING RATIONAL NUMBERS WITH REGIONS

Exploration

1. Figure A represents a unit region. It is separated into 3 smaller congruent regions and 2 of these regions are shaded.



A

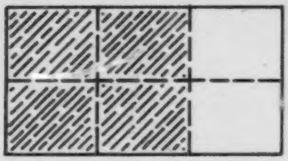
a. What is the measure of the shaded region? $\left(\frac{2}{3}\right)$

b. How are the 2 and the 3 in the fraction $\frac{2}{3}$ related to the unit region? (Unit region is separated into 3 congruent regions. 2 of these are shaded.)

2. Figure B represents the same unit region. It is separated into 6 smaller congruent regions and 4 of these regions are shaded.

a. What is the measure of the shaded region? $\left(\frac{4}{6}\right)$

b. How are the 4 and 6 in the fraction $\frac{4}{6}$ related to the unit region? (Unit segment is separated into 6 congruent segments. 4 of these are shaded.)

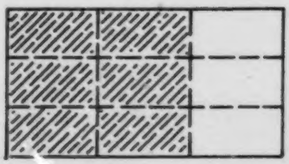


B

3. Trace a rectangle congruent to figure A. Draw broken lines to separate the region into 9 congruent regions and shade 6 of these regions as shown in figure C.

a. What is the measure of the shaded region? $\left(\frac{6}{9}\right)$

b. How are the 6 and the 9 of the fraction $\frac{6}{9}$ related to the unit region? (Unit region is separated into 9 congruent segments. 6 are shaded.)



C

4. a. Is the unit region the same size in figures A, B, and C ? (yes)

b. Are the shaded regions of A, B, and C congruent? (yes)

c. Are the rational numbers $\frac{2}{3}$, $\frac{4}{6}$, and $\frac{6}{9}$ the measures of these congruent shaded regions? (yes)

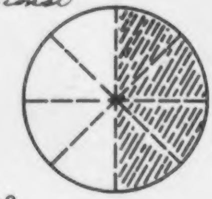
d. Are $\frac{2}{3}$, $\frac{4}{6}$, and $\frac{6}{9}$ names for the same rational number? (yes)

5. Draw 3 congruent rectangular regions. Label them A, B, and C.

a. Separate A into 3 smaller congruent regions. Shade 1 region. What is the measure of the shaded region? ($\frac{1}{3}$)

b. Separate B into 6 smaller congruent regions and shade 2 of them. What is the measure of the shaded region? ($\frac{2}{6}$)

c. Separate C into 9 congruent regions. Shade 3 of these regions. Are $\frac{1}{3}$, $\frac{2}{6}$, and $\frac{3}{9}$ names for the same rational number? (yes) Why? (They name the measures of congruent regions)



6. The measure of the circular region at the right is 1. The measure of the shaded part is $\frac{4}{8}$. What is the relation of the 4 and 8 of the fraction to the unit region? (The unit region is separated into 8 congruent regions. 4 regions are shaded.)

7. Trace the circular region above. Separate it into 2 congruent regions. Shade 1 part.

a. What is the measure of the shaded region? ($\frac{1}{2}$)

b. Are $\frac{4}{8}$ and $\frac{1}{2}$ names for the same number since they name the measures of congruent regions? (yes)

8. Draw a circular region congruent to the one in exercise 6. Separate it into 4 congruent regions and shade 2 of the parts. Is $\frac{2}{4}$ another name for $\frac{1}{2}$ and $\frac{4}{8}$? (yes) Why? (They name the measures of congruent regions)

A rational number may have many different names.

$$1 = \frac{2}{2} = \frac{4}{4} = \frac{8}{8}$$

$$\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$$

$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9}$$

Exercise Set 4

1. Answer these questions for each figure below.
   a. Into how many congruent regions is the unit region separated?
   b. How many congruent regions are shaded?
   c. What fraction name best describes the measure of the shaded region?

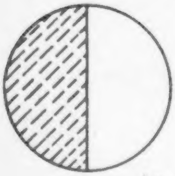


Figure 1 (a. 2 b. 1 c. $\frac{1}{2}$)

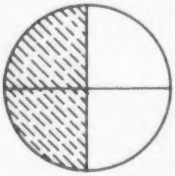


Figure 2 (a. 4 b. 2 c. $\frac{2}{4}$)

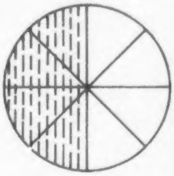


Figure 3 (a. 8 b. 4 c. $\frac{4}{8}$)

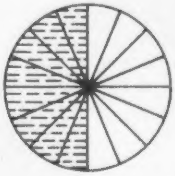


Figure 4 (a. 16 b. 8 c. $\frac{8}{16}$)

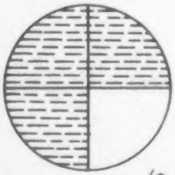


Figure 5 (a. 4 b. 3 c. $\frac{3}{4}$)

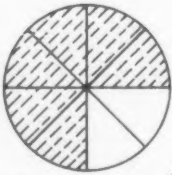


Figure 6 (a. 8 b. 6 c. $\frac{6}{8}$)

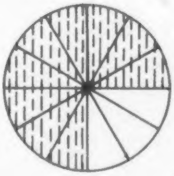


Figure 7 (a. 12 b. 9 c. $\frac{9}{12}$)

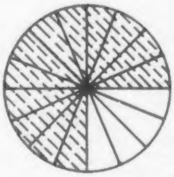


Figure 8 (a. 16 b. 12 c. $\frac{12}{16}$)

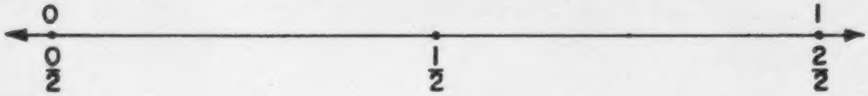
2. Do the fractions in exercise c for figures 1 through 4 name the same rational number? (yes) What rational number do they name? $\left(\frac{1}{2}\right)$

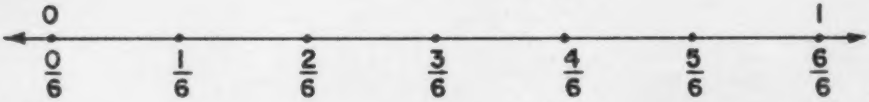
3. Do the fractions in exercise c for figures 5 through 8 name the same rational number? (yes) What rational number do they name? $\left(\frac{3}{4}\right)$

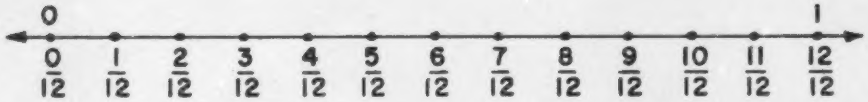
4. Write three other names for $\frac{1}{2}$. $\left(\frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \frac{6}{12}, \frac{7}{14}, \text{etc.}\right)$

5. Write three other names for $\frac{3}{4}$. $\left(\frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \frac{15}{20}, \frac{18}{24}, \text{etc.}\right)$

6. Write three names for $\frac{1}{4}$. $\left(\frac{2}{8}, \frac{4}{16}, \frac{3}{12}, \frac{5}{20}, \frac{6}{24}, \text{etc.}\right)$

A


B


C


7. Look at the number lines A, B, and C above. Three congruent segments, each having the measure of 1, have been separated into smaller congruent segments. Answer the following questions about each number line.

   a. Into how many congruent segments has the unit segment been separated? (a. 2 b. 6 c. 12)

   b. What fraction best names the measure of each smaller congruent segment? (a. $\frac{1}{2}$; b. $\frac{1}{6}$; c. $\frac{1}{12}$)

8. Make true statements by writing a different fraction in each space.

a. $\frac{1}{2} = \underline{\left(\frac{2}{4}\right)} = \underline{\left(\frac{3}{6}\right)} = \underline{\left(\frac{4}{8}\right)}$

b. $\frac{1}{3} = \underline{\left(\frac{2}{6}\right)} = \underline{\left(\frac{3}{9}\right)}$

c. $\frac{2}{3} = \underline{\left(\frac{4}{6}\right)} = \underline{\left(\frac{6}{9}\right)}$

d. $\frac{1}{4} = \underline{\left(\frac{2}{8}\right)} = \underline{\left(\frac{3}{12}\right)} = \underline{\left(\frac{4}{16}\right)}$

e. $\frac{3}{4} = \underline{\left(\frac{6}{8}\right)} = \underline{\left(\frac{9}{12}\right)} = \underline{\left(\frac{12}{16}\right)}$

f. $1 = \underline{\left(\frac{2}{2}\right)} = \underline{\left(\frac{3}{3}\right)} = \underline{\left(\frac{4}{4}\right)}$

RATIONAL NUMBERS WITH SETS OF OBJECTS

Objectives: To develop the idea that a rational number shows the relation of a subset to a whole set.

To use sets of objects to help children discover different names for rational numbers.

Materials: Sets of objects, arrays

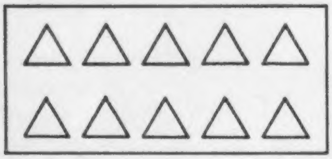
Vocabulary: Set, subset

Suggested Teaching Procedure

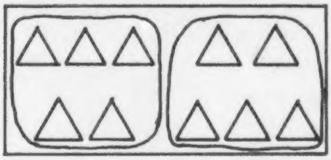
Work through Exploration together. Follow up with pictures of arrays or sets as shown in section "Materials", or better yet, concrete objects. Use sets of 12 objects, and have the pupils show fourths of the set, thirds of the set, sixths of the set, etc. Also, have them tell what rational number tells what part a certain subset is of the whole set. Sets, or pictures of sets, with 16 objects (or some other number) may be used in a similar manner.

RATIONAL NUMBERS WITH SETS OF OBJECTS

Exploration



A

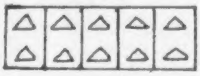


B

1. Figure A shows a picture of a set of 10 objects. What rational number best describes what part of the set each object is? $(\frac{1}{10})$

2. Figure B shows the same set separated into subsets each having the same number of objects.
   a. Into how many subsets has the set shown in A been separated in B ? (2)
   b. How many objects are in each subset? (5)
   c. What part of the set is in each subset? ($\frac{1}{2}$ or $\frac{5}{10}$)
   d. Do $\frac{1}{2}$ and $\frac{5}{10}$ name the same rational number? (yes)

3. Trace figure A. Separate the objects into 5 subsets, each having the same number of objects.



(or any other separation with 2 in each subset)

Replace n by a number which makes each sentence true.

a. $\frac{1}{5} = \frac{n}{10}$ (2)

c. $\frac{3}{5} = \frac{n}{10}$ (6)

b. $\frac{2}{5} = \frac{n}{10}$ (4)

d. $\frac{4}{5} = \frac{n}{10}$ (8)

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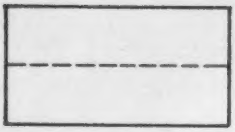
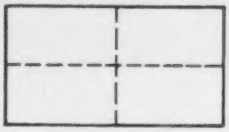
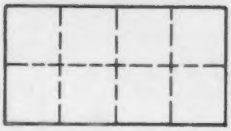
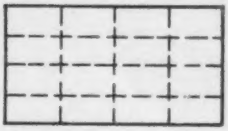
Exercise Set 5


A


B


C


D

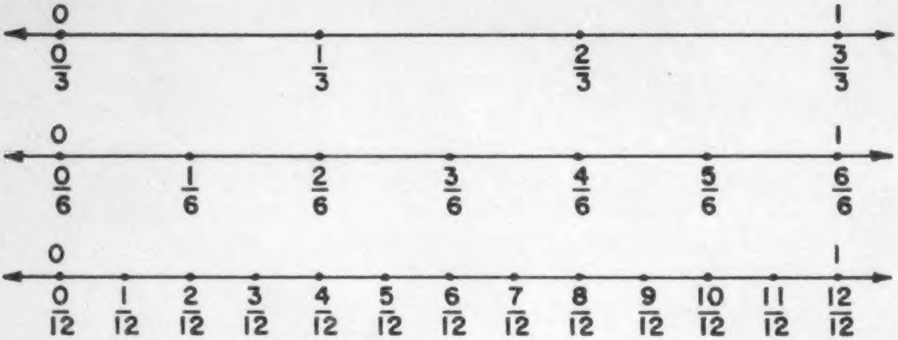
A, B, C, and D are congruent rectangular regions. The measure of each is 1. Use them to help you answer the following exercises.

1. There are 4 quarts in a gallon. 3 quarts is the same amount as
   a. (3) fourths of a gallon.
   b. (6) eighths of a gallon.
   c. (12) sixteenths of a gallon.

2. There are 16 ounces in a pound. 8 ounces is the same amount as
   a. 1 (half) of a pound.
   b. 2 (fourths) of a pound.
   c. 4 (eighths) of a pound.
   d. 8 (sixteenths)) ____ of a pound.

3. How many quarts are there in $\frac{1}{2}$ gallon? (2)

4. How many ounces are there in $\frac{1}{4}$ pound? (4)


0
1
0/3
1/3
2/3
3/3
0
1
0/6
1/6
2/6
3/6
4/6
5/6
6/6
0
1
0/12
1/12
2/12
3/12
4/12
5/12
6/12
7/12
8/12
9/12
10/12
11/12
12/12

Use the number lines above to answer the following exercises.

5. 8 inches may be written

   a. (8) twelfths of a foot.

   b. (4) sixths of a foot.

   c. (2) thirds of a foot.

6. Ten months could be written as

   a. 10 (twelfths) of a year.

   b. 5 (sixths) of a year.

7. What part of a dozen cookies are

   a. 6 cookies? ($\frac{1}{2}$ or $\frac{6}{12}$)

   b. 4 cookies? ($\frac{4}{12}$ or $\frac{1}{3}$)

   c. 8 cookies? ($\frac{8}{12}$ or $\frac{2}{3}$)

   d. 10 cookies? ($\frac{10}{12}$ or $\frac{5}{6}$)

8. What part of a yard is

   a. 1 foot? ($\frac{1}{3}$)

   b. 2 feet? ($\frac{2}{3}$)

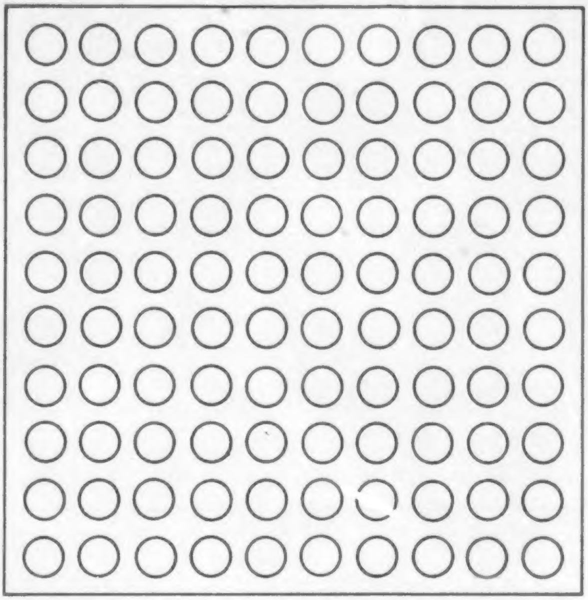
9. What part of a year is

   a. 4 months? ($\frac{4}{12}$ or $\frac{1}{3}$)

   b. 6 months? ($\frac{6}{12}$ or $\frac{1}{2}$)

10. How many inches are there in $\frac{1}{3}$ of a foot? (4)

11. How many eggs are there in $\frac{1}{6}$ of a dozen? (2)



12. Above is a picture of a set of 100 pennies. Use it to answer the following questions. Write the fraction with the smallest denominator for each rational number used in your answers.

What part of one dollar is

a. 50 pennies? $(\frac{1}{2})$

b. 10 pennies? $(\frac{1}{10})$

c. 25 pennies? $(\frac{1}{4})$

d. 5 pennies? $(\frac{1}{20})$

e. 20 pennies? $(\frac{1}{5})$

THE SIMPLEST FRACTION NAME FOR A RATIONAL NUMBER

Objectives: To review

(a) many names for one rational number.
(b) meaning of numerator and denominator.

To generalize

(a) any rational number may be renamed by multiplying numerator and denominator of a fraction name by the same counting number greater than 1.
(b) although there are more names for a rational number than can be counted, the simplest name for a rational number is that fraction in which numerator and denominator have no common factors except 1.
(c) to find the simplest name, remove greatest common factor of numerator and denominator.

Vocabulary: Simplest name, prime number, common factor, greatest common factor, digit, odd, prime factor, complete factorization, composite

Suggested Teaching Procedure

Brief review of Chapter 2, Grade 5 "Factors and Primes" recommended if teacher feels it is needed.

Have class read and discuss Exploration on "Simplest Name for a Rational Number". It is hoped children will discover intuitively how to find many names for any rational number. You will have to judge whether or not your pupils are ready to express the generalization: to find another name for a rational number, multiply the numerator and denominator by the same counting number. When this has been done, proceed with rest of Exploration and Exercise Sets 6 and 7.

THE SIMPLEST FRACTION NAME FOR A RATIONAL NUMBER

Exploration

We have found that many fractions name one rational number. For example, we know that

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8}.$$

1. Find three more fractions that belong on this list. $(\frac{5}{10}, \frac{6}{12}, \frac{7}{14}, etc.)$

2. Do these fractions belong on the list? (yes)

$$\frac{50}{100}, \frac{100}{200}, \frac{111}{222}.$$

3. Find n , m , and p so that each fraction names $\frac{1}{2}$.

$$\frac{1}{2} = \frac{n}{14}\,(7), \quad \frac{1}{2} = \frac{8}{m}\,(16), \quad \frac{1}{2} = \frac{p}{250}\,(125)$$

We have also found several names for $\frac{2}{3}$.

$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{8}{12}.$$

4. Do these fractions belong on this list of names of $\frac{2}{3}$? (yes)

$$\frac{20}{30}, \frac{2 \times 4}{3 \times 4}, \frac{14}{21}, \frac{50}{75}, \frac{2 \times 876}{3 \times 876}, \frac{2 \times 2 \times 5 \times 7}{2 \times 3 \times 5 \times 7}$$

5. Suppose that m and n are counting numbers. Give three other names for $\frac{m}{n}$.

$$\left(\frac{m \times 2}{n \times 2}\right) \quad \left(\frac{m \times 3}{n \times 3}\right) \quad \left(\frac{m \times 4}{n \times 4}\right) \text{ etc.}$$

Now try to imagine the set of all fractions which name one rational number:

$$\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10} \ldots\}$$

$$\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12} \ldots\}$$

$$\{\frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16} \ldots\}$$

Each such set contains a fraction with a denominator smaller than the rest. We will call this fraction the simplest fraction name for the rational number. It is a name we often use.

Any other fraction in each set can be found from the simplest fraction name. Do you know the rule for finding the other fractions?

(Any rational number may be renamed by multiplying numerator and denominator of a fraction name by the same counting number greater than 1.)

FINDING THE SIMPLEST FRACTION NAME

Exploration

How can you tell whether a fraction is the simplest name for a rational number? There are no common factors (in numerator or denominator) Which ones of these are simplest fraction names? $\left(\frac{5}{11}, \frac{9}{14}, \frac{3 \times 5 \times 5 \times 7}{2 \times 2 \times 11}\right)$

$$\frac{3}{12}, \frac{8}{10}, \frac{5}{11}, \frac{9}{14}, \frac{6}{2}, \frac{3 \times 5 \times 5 \times 7}{2 \times 2 \times 11}, \frac{2 \times 5 \times 7 \times 11}{3 \times 5 \times 13}$$

Is $\frac{2 \times 8765}{2 \times 3341}$ a simplest fraction name? (No, because 2 is a factor of both numerator and denominator)

Perhaps you remember that 13 is a prime number. Is $\frac{13}{6893}$ a simplest fraction name? (Yes, because 97 is not a factor of 6893, and no other number is a factor of 97.)

Is $\frac{n}{13}$ always a simplest fraction name if $n < 13$? (Yes)

These examples should suggest two things to you:

First, a simplest fraction name is one in which the numerator and denominator have no common factors except 1

Second, you can find the simplest fraction name from any fraction in the set by finding the greatest common factor of its numerator and denominator.

Here are several examples showing how you can find simplest fraction names.

1) Find the simplest fraction name for $\frac{30}{45}$. $\left(\frac{2}{3}\right)$ First factor numerator and denominator completely.

$$\frac{30}{45} = \frac{2 \times 3 \times 5}{3 \times 3 \times 5}$$

Next remove the common prime factors shown (3 and 5).

$$\frac{30}{45} = \frac{2 \times 3 \times 5}{3 \times 3 \times 5} = \frac{2}{3}.$$

2) Find the simplest fraction name for $\frac{96}{375}$. $\left(\frac{32}{125}\right)$

Here is a different method. First test 2 as a factor of both numerator and denominator. You find that 2 is not a factor of 375 because the units digit is odd. This means that 2 cannot be a common factor of 96 and 375.

Now test 3 as a factor of numerator and denominator You find:

$$\frac{96}{375} = \frac{3 \times 32}{3 \times 125}.$$

Now remove the common factor.

$$\frac{96}{375} = \frac{\cancel{3} \times 32}{\cancel{3} \times 125} = \frac{32}{125}.$$

Next test 3 again as a factor of 32 and 125. Since 3 is not a factor of 32, it is not a common factor. Continue and try 5 as a common factor. Notice, however, that 2 is the only prime factor of 32 and that 5 is the only prime factor of 125. If you see this, it will save you time because you know right away that 32 and 125 have greatest common factor 1. This means that $\frac{32}{125}$ is the simplest fraction name for $\frac{96}{375}$.

3) Now try the method used in example 1 to find the simplest fraction name for $\frac{90}{84}$. $\left(\frac{15}{14}\right)$

4) Next try the method used in example 2 to find the simplest fraction name for $\frac{108}{100}$. $\left(\frac{27}{25}\right)$

If you know many multiplication facts you can often shorten the work in finding simplest fraction names. For example, in finding the simplest fraction name for $\frac{56}{88}$ you might remember that $8 \times 7 = 56$ and $8 \times 11 = 88$. Then you can write

$$\frac{56}{88} = \frac{8 \times 7}{8 \times 11} = \frac{7}{11} .$$

5) How can you use the fact: $12 \times 12 = 144$ in finding the simplest fraction name for $\frac{60}{144}$? $\left(\frac{5 \times 12}{12 \times 12} = \frac{5}{12}\right)$

Of course you can always use one of the methods shown in the examples.

Exercise Set 6

1. Write three other fractions naming each of the following numbers.

   a. $\frac{1}{3}$ $\left(\frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \text{etc.}\right)$

   b. $\frac{2}{5}$ $\left(\frac{4}{10}, \frac{8}{20}, \frac{16}{40}, \text{etc.}\right)$

   c. $\frac{5}{4}$ $\left(\frac{10}{8}, \frac{15}{12}, \frac{20}{16}, \text{etc.}\right)$

   d. $\frac{3}{4}$ $\left(\frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \text{etc.}\right)$

   e. $\frac{2}{3}$ $\left(\frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \text{etc.}\right)$

   f. $\frac{7}{8}$ $\left(\frac{14}{16}, \frac{21}{24}, \frac{28}{32}, \text{etc.}\right)$

   g. $\frac{1}{6}$ $\left(\frac{2}{12}, \frac{3}{18}, \frac{4}{24}, \text{etc.}\right)$

   h. $\frac{3}{2}$ $\left(\frac{6}{4}, \frac{9}{6}, \frac{12}{8}, \text{etc.}\right)$

   i. $\frac{3}{5}$ $\left(\frac{6}{10}, \frac{9}{15}, \frac{12}{20}, \text{etc.}\right)$

2. Copy the fractions which are simplest fraction names.

   a. $\frac{6}{12}, \frac{7}{12}, \frac{8}{12}, \frac{9}{12}, \frac{10}{12}, \frac{11}{12}$ $\left(\frac{7}{12}, \frac{11}{12}\right)$

   b. $\frac{2}{3}, \frac{11}{4}, \frac{5}{3}, \frac{3}{4}, \frac{4}{6}, \frac{5}{8}$ $\left(\frac{2}{3}, \frac{11}{4}, \frac{5}{3}, \frac{3}{4}, \frac{5}{8}\right)$

   c. $\frac{7}{8}, \frac{8}{8}, \frac{6}{8}, \frac{2}{8}, \frac{1}{8}, \frac{3}{8}$ $\left(\frac{7}{8}, \frac{1}{8}, \frac{3}{8}\right)$

   d. $\frac{7}{3}, \frac{7}{16}, \frac{7}{8}, \frac{7}{5}, \frac{7}{10}, \frac{7}{12}$ $\left(\frac{7}{3}, \frac{7}{16}, \frac{7}{8}, \frac{7}{5}, \frac{7}{10}, \frac{7}{12}\right)$

3. Complete by supplying the missing numerator or denominator,

a. $\frac{3}{6} = \frac{(12)}{24}$

b. $\frac{3}{15} = \frac{12}{(60)}$

c. $\frac{7}{12} = \frac{(5 \times 7)}{5 \times 12}$

d. $\frac{9}{18} = \frac{54}{(108)}$

e. $\frac{6}{24} = \frac{7 \times 6}{(7 \times 24)}$

f. $\frac{5}{25} = \frac{(30)}{150}$

4. Use complete factorization to find simplest fraction names for:

a. $\frac{72}{81} \left(\frac{2 \times 2 \times 2 \times 3 \times 3}{3 \times 3 \times 3 \times 3} = \frac{8}{9}\right)$ b. $\frac{84}{105} \left(\frac{3 \times 2 \times 2 \times 7}{3 \times 5 \times 7} = \frac{4}{5}\right)$ c. $\frac{98}{196} \left(\frac{2 \times 7 \times 7 \times 1}{2 \times 2 \times 7 \times 7} = \frac{1}{2}\right)$

5. Find the simplest fraction names for the following. You should be able to do this using multiplication facts only.

a. $\frac{6}{9}$ $\left(\frac{2}{3}\right)$

b. $\frac{10}{15}$ $\left(\frac{2}{3}\right)$

c. $\frac{4}{8}$ $\left(\frac{1}{2}\right)$

d. $\frac{9}{12}$ $\left(\frac{3}{4}\right)$

e. $\frac{12}{8}$ $\left(\frac{3}{2}\right)$

f. $\frac{5}{15}$ $\left(\frac{1}{3}\right)$

g. $\frac{12}{16}$ $\left(\frac{3}{4}\right)$

h. $\frac{21}{24}$ $\left(\frac{7}{8}\right)$

i. $\frac{4}{10}$ $\left(\frac{2}{5}\right)$

j. $\frac{4}{16}$ $\left(\frac{1}{4}\right)$

k. $\frac{6}{12}$ $\left(\frac{1}{2}\right)$

l. $\frac{8}{16}$ $\left(\frac{1}{2}\right)$

m. $\frac{8}{12}$ $\left(\frac{2}{3}\right)$

n. $\frac{7}{21}$ $\left(\frac{1}{3}\right)$

o. $\frac{10}{25}$ $\left(\frac{2}{5}\right)$

p. $\frac{15}{20}$ $\left(\frac{3}{4}\right)$

q. $\frac{12}{15}$ $\left(\frac{4}{5}\right)$

r. $\frac{77}{88}$ $\left(\frac{7}{8}\right)$

s. $\frac{16}{12}$ $\left(\frac{4}{3}\right)$

t. $\frac{20}{24}$ $\left(\frac{5}{6}\right)$

u. $\frac{25}{20}$ $\left(\frac{5}{4}\right)$

6. Find the simplest fraction name for each number. Then use $<$, $>$, or $=$ in each blank to make a true statement.

a. $\frac{9}{15}$ (<) $\frac{20}{25}$

b. $\frac{8}{10}$ (>) $\frac{9}{15}$

c. $\frac{12}{16}$ (=) $\frac{36}{48}$

d. $\frac{12}{22}$ (>) $\frac{10}{33}$

e. $\frac{3}{27}$ (<) $\frac{8}{18}$

f. $\frac{8}{14}$ (=) $\frac{20}{35}$

Exercise Set 7

1. Make true statements by filling in the blanks.

a. $\frac{2}{3} = \frac{(18)}{27}$

b. $\frac{5}{4} = \frac{35}{(28)}$

c. $\frac{4}{7} = \frac{4 \times 16}{(4 \times 7)}$

d. $\frac{(5)}{8} = \frac{45}{72}$

e. $\frac{9}{(5)} = \frac{63}{35}$

f. $\frac{1}{2} = \frac{(1 \times n)}{2 \times n}$

g. $\frac{20}{40} = \frac{27}{(54)}$

h. $\frac{25}{(75)} = \frac{16}{48}$

2. a. The measure of $\frac{2}{3}$ of a foot in inches is (8).

b. The measure of $\frac{1}{2}$ yard in inches is (18).

c. The measure of $\frac{3}{4}$ hours in minutes is (45).

d. Twenty minutes is $\frac{(1)}{3}$ of an hour.

e. (4) weeks is $\frac{1}{13}$ of a year.

3. Write "prime" if the number is prime. Name at least one prime factor if the number is composite.

Example: 73 prime.

Neither 2, nor 3, nor 5 is a factor. (Do you remember how to tell?) By division we find that 7 is not a factor. This is enough to show that 73 is a prime. (Why?)

a. 58 (2 or 29)

b. 97 (prime)

c. 51 (3 or 17)

d. 365 (5 or 73)

e. 705 (3, 5, or 47)

f. 91 (7 or 13)

g. 5280 (2, 3, 5, or 11)

h. 143 (11 or 13)

4. Use complete factorization to find the simplest fraction name.

   a. $\frac{45}{180}$ $\left(\frac{3 \times 3 \times 5}{2 \times 2 \times 3 \times 3 \times 5} = \frac{1}{4}\right)$ d. $\frac{105}{143}$ $\left(\frac{3 \times 5 \times 7}{11 \times 13} = \frac{105}{143}\right)$

   b. $\frac{126}{60}$ $\left(\frac{2 \times 3 \times 3 \times 7}{2 \times 2 \times 3 \times 5} = \frac{21}{10}\right)$ e. $\frac{90}{300}$ $\left(\frac{2 \times 3 \times 3 \times 5}{2 \times 2 \times 3 \times 5 \times 5} = \frac{3}{10}\right)$

   c. $\frac{97}{365}$ $\left(\frac{1 \times 97}{5 \times 73} = \frac{97}{365}\right)$ f. $\frac{10 \times 9 \times 11}{6 \times 21 \times 11}$ $\left(\frac{2 \times 5 \times 3 \times 3 \times 11}{2 \times 3 \times 3 \times 7 \times 11} = \frac{5}{7}\right)$

5. By finding the simplest fraction name for each of these numbers, tell which is greater.

   a. $\frac{24}{42}$ or $\frac{40}{56}$ $\left(\frac{40}{56}\right)$ b. $\frac{16}{44}$ or $\frac{45}{99}$ $\left(\frac{45}{99}\right)$ c. $\frac{81}{117}$ or $\frac{72}{104}$ (same)

6. In Jefferson school there were 325 pupils in all and 175 girls. In Washington school there were 312 pupils in all and 144 girls. In which school do girls form the larger part? (Jefferson) In which school or schools are more than $\frac{1}{2}$ of the pupils girls? (Jefferson)

7. Find the simplest fraction name for:

   a. the measure in feet of 16 inches. $\left(\frac{4}{3}\right)$

   b. the measure in days of 33 hours. $\left(\frac{11}{8}\right)$

   c. the measure in miles of 440 yards. $\left(\frac{1}{4}\right)$

   d. the measure in pounds of 20 ounces. $\left(\frac{5}{4}\right)$

   e. the measure in hours of 45 minutes. $\left(\frac{3}{4}\right)$

8. Is this true? $\frac{3 + 8}{4 + 8} = \frac{3}{4}$? Why? (No, because to rename a fraction you must multiply numerator and denominator by same number)

BRAINTWISTER

9. Is $\frac{3843}{10{,}000}$ a simplest fraction name? (yes) You do not need to make any long computation to find the answer. (Because the only factors of denominator are 2 and 5. Neither 2 nor 5 is a factor of 3843.)

COMMON DENOMINATOR
COMMON MULTIPLE
LEAST COMMON MULTIPLE

Objectives: To show a need for renaming rational numbers by fractions with the same denominator.

To show that the set of possible common denominators for two fractions is the same as the set of common multiples of their denominators.

To develop a method for finding the least common multiple of two numbers by using their complete factorizations.

Vocabulary: Common denominator, multiple, common multiple, set of multiples, intersection, ($\cap$), compare

Suggested Teaching Procedure

The pupils know that any rational number has many fraction names. In the previous lesson they were concerned with finding simplest fraction names. However, the simplest fraction name is frequently not the name most useful for answering certain questions, e.g., deciding which of two fractions names the larger number. It is therefore useful to develop a method for finding common denominators for any two fractions. In the next three sections, the problem of finding common denominators is related to the problem of finding common multiples for the denominators of the fractions.

In the sections, "Common Denominator", "Common Multiple", and "Least Common Multiple", use is made of the ideas and techniques developed in the unit, "Factors and Primes." Some review of this unit may be needed.

Teacher and pupils should work through each Exploration together. Opportunity should be given for use of the new vocabulary. The diagram proposed in the section "Least Common Multiple" for recording the prime factorizations of the two numbers for which the least common multiple is to be found will be particularly helpful to the pupils.

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COMMON DENOMINATOR

Sometimes the simplest fraction name is not the name you need to solve a problem. Suppose that you want to know which is larger, $\frac{2}{3}$ of a mile or $\frac{7}{10}$ of a mile. Would you prefer to have 2 of 3 equal stacks of pennies or to separate the same set of pennies into 10 equal stacks and take 7?

In either case you want to know:

Which is greater, $\frac{2}{3}$ or $\frac{7}{10}$? $\left(\frac{7}{10}\right)$

Both names are simplest, but we cannot answer the question with them. Here is another example that may help you to find the answer.

Which is greater, $\frac{1}{2}$ or $\frac{7}{10}$?

You know that $\frac{1}{2} = \frac{5}{10}$. You know that $\frac{7}{10} > \frac{5}{10}$. So you can say

$$\frac{1}{2} < \frac{7}{10}.$$

Which is greater $\frac{2}{3}$ or $\frac{5}{9}$? $\left(\frac{2}{3}\right)$

Which is greater $\frac{2}{6}$ or $\frac{9}{24}$? $\left(\frac{9}{24}\right)$

Which is greater $\frac{17}{8}$ or 2? $\left(\frac{17}{8}\right)$

The trick is to find for each number names with the same denominator. Think again about $\frac{2}{3}$ and $\frac{7}{10}$. What other denominators do fractions naming $\frac{7}{10}$ have? What other denominators do fractions naming $\frac{2}{3}$ have? What is the smallest number which is in both lists of denominators?

The answers to these questions help you to see that:

$$\frac{2}{3} = \frac{20}{30} \text{ and } \frac{7}{10} = \frac{21}{30}.$$

You know then that $\frac{7}{10} > \frac{2}{3}$.

You could answer your question about $\frac{2}{3}$ and $\frac{7}{10}$ as soon as you knew that 30 was a denominator both for $\frac{2}{3}(\frac{20}{30})$ and $\frac{7}{10}(\frac{21}{30})$.

The set of denominators for $\frac{2}{3}$ is $\{3, 6, 9, 12, . . .\} = K$

The set of denominators for $\frac{7}{10}$ is $\{10, 20, 30, . . .\} = L$

Set K is called the set of multiples of 3.

Set L is called the set of multiples of 10.

The numbers common to both sets are called the common multiples of 3 and 10.

The numbers both sets have in common are also the numbers you can use as denominators for both $\frac{2}{3}$ and $\frac{7}{10}$. They are called common denominators for $\frac{2}{3}$ and $\frac{7}{10}$. Before you study fractions any further, you should find out more about common multiples.

COMMON MULTIPLE

We use the word "multiple" as another way to talk about factors. Instead of saying

4 is a factor of 12

we may say

12 is a multiple of 4.

This idea is not strange. Instead of saying

3 is less than 5

we might say

5 is greater than 3.

Instead of saying

John is younger than Bruce

we might say

Bruce is older than John.

What is the other way of saying these?

I am taller than you.

Today is warmer than yesterday.

The relation between factor and multiple is another example of the same idea. Put these statements into the language of multiples:

7 is a factor of 21 (21 is a multiple of 7)

3 is not a factor of 31 (31 is not a multiple of 3)

12 is a factor of 12. (12 is a multiple of 12)

Put these into the language of factors:

14 is a multiple of 7. (7 is a factor of 14)

12 is a multiple of 12. (12 is a factor of 12)

18 is a multiple of both 9 and 2. (9 and 2 are both factors of 18)

Because 18 is a multiple of both 9 and 2, 18 is called a common multiple of 9 and 2. Because 12 = 3 × 4 and 12 = 2 × 6, 12 is a common multiple of 4 and 6. Is 12 a common multiple of 3 and 4 ? (yes) Of 2 and 3 ? (yes) Of 4 and 12? (yes)

A good way to think about common multiples is to use the language of sets.

Let R be the set of all multiples of 4, and let S be the set of all multiples of 3.

$$R = \{4, 8, 12, 16, 20, 24, 28, \ldots\}$$

$$S = \{3, 6, 9, 12, 15, 18, 21, 24, 27, \ldots\}$$

The set of common multiples of 3 and 4 is

$$R \cap S = \{12, 24, \ldots\}.$$

Exercise Set 8

1. Below are pairs of numbers. Show the set of multiples of each number. Then show the set of common multiples.

   Example: 3, 5

   A = set of multiples of 3 = {3, 6, 9, 12, 15, 18 ...}

   B = set of multiples of 5 = {5, 10, 15, 20 ...}

   $A \cap B$ = set of common multiples of 3 and 5 = {15, 30 ...}

   a. 4 , 6 (4, 8, 12, 16, 20, 24, ...) (6, 12, 18, 24, ...) (12, 24, 48, ...)
   b. 6 , 8 (6, 12, 18, 24, ...) (8, 16, 24, 32, ...) (24, 48, 72, ...)
   c. 15 , 10 (15, 30, 45, ...) (10, 20, 30, ...) (30, 60, 90, ...)
   d. 9 , 6 (9, 18, 27, ...) (6, 12, 18, ...) (18, 36, 54, ...)
   e. 10, 20 (10, 20, 30, 40, ...) (20, 40, 60, ...) (40, 80, 120, ...)

2. In the example in exercise 1, is 45 a common multiple of 3 and 5 ? (yes) Is 60 ? (yes) If n is a counting number, is n × 15 always a common multiple of 3 and 5? (yes)

3. The product of two numbers is always a multiple common to both numbers. Is it *ever* the smallest of all common multiples? (yes) Is it *always* the smallest? (No) Give examples. (Smallest multiple of 2 and 3 *is* 2×3 or 6. 4×6 *is* *not* smallest multiple of 4 and 6.)

4. I am thinking of two numbers. They have 18 as a common multiple.

   a. Is 36 a common multiple of the two numbers? (yes)
   b. If n is a counting number, is 18 × n always a common multiple of the two numbers? (yes)
   c. Could 9 be a common multiple of the two numbers? (yes) Give an example if there is one. (3, 9)

LEAST COMMON MULTIPLE

There are two things which seem to be true about the set of all the common multiples of any two numbers.

1) Every multiple of the smallest common multiple is also a common multiple.

2) No other numbers are common multiples. For example, the set of common multiples of 2 and 3 begins

$$\{6, 12, 18, 24, \ldots\}$$

It seems to consist of only the multiples of 6. 6 is the smallest common multiple of 2 and 3.

Because 1) and 2) are always true, we only have to know the smallest common multiple, then we can find all common multiples. The smallest common multiple is usually called the least common multiple.

Exercise Set 9

1. The least common multiple of two numbers is 10. What are the other common multiples? (20, 30, 40, ...)

2. Find two different pairs of numbers with least common multiple 18. (2 and 9; 6 and 9; 6 and 18; etc.)

3. Express this idea in factor language:

   The least common multiple of 3 and 4 is 12.
   (4 is a factor of 2; 3 is a factor of 12. 12 is the smallest number with both 3 and 4 as factors.)

4. If you want to compare $\frac{5}{6}$ to $\frac{7}{9}$ what is the smallest denominator you could use? (18)

5. Find several members of the set of multiples for each number below. Underline the least common multiple for each pair.

a. 9 , 5 (9, 18, 27, 36, 45, ...) (5, 10, 15, 20, 25, 30, 35, 40, 45, ...)
c. 5 , 6 (5, 10, 15, 20, 25, 30, ...) (6, 12, 18, 24, 30, ...)

b. 7 , 8 (7, 14, 21, 28, 35, 42, 49, 56, ...) (8, 16, 24, 32, 40, 48, 56, ...)
d. 11 , 7 (11, 22, 33, 44, 56, 66, 77, ...) (7, 14, 28, 35, 42, 49, 56, 63, 70, 77, ...)

6. Find the least common multiple of each pair. Then show three of the set of all common multiples.

a. 12 , 13 (156, 312, 468, ...)
c. 21 , 12 (84, 168, 252, ...)

b. 5 , 8 (40, 80, 120, ...)
d. 17 , 5 (85, 170, 255, ...)

Exploration

Until now we have found the least common multiple (l.c.m.) of two numbers by listing multiples of each number. But this may be a long process even if the numbers are small. For example, to find the l.c.m. of 8 and 9 we find:

Set of multiples of 8 = {8, 16, 24, 32, 40, 48, 56, 64, 72 ...}

Set of multiples of 9 = {9, 18, 27, 36, 45, 54, 63, 72 ...}

It would be even harder to test our belief that the set of common multiples of 8 and 9 is

{72, 144, 216, 288, ...} .

There is a much easier way to do both.

A. First we factor the numbers completely:

$$8 = 2 \times 2 \times 2.$$

$$9 = 3 \times 3.$$

Suppose that n is any common multiple of 8 and 9. Think about the expression for n as a product of primes. Since n is a multiple of 8, $2 \times 2 \times 2$ must be a piece of this expression.

$$n = 2 \times 2 \times 2 \times (\text{any other prime factors}).$$

Since n is a multiple of 9, 3×3 must also be a piece of this expression.

$$n = 3 \times 3 \times (\text{any other prime factors}).$$

We know then that

$$n = 2 \times 2 \times 2 \times 3 \times 3 \times (\text{any other prime factors}).$$

If n is the least common multiple then

$$n = 2 \times 2 \times 2 \times 3 \times 3 = 8 \times 9 = 72.$$

Any other common multiple can be expressed as

$$8 \times 9 \times (\text{other factors}).$$

This shows that every other common multiple of 8 and 9 is a multiple of 72.

B. Here is another example: Find the l.c.m. of 60 and 270.

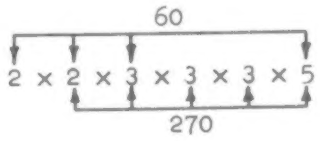
$$60 = 2 \times 2 \times 3 \times 5.$$

$$270 = 2 \times 3 \times 3 \times 3 \times 5.$$

The l.c.m. must have at least two 2's, three 3's, and one 5 in its factorization. So

$$\text{l.c.m. of } 60 \text{ and } 270 = 2 \times 2 \times 3 \times 3 \times 3 \times 5$$
$$= 540$$

We can think of the l.c.m. in this way



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We have put in all the prime factors we need to get a multiple of 270 and a multiple of 60. We get the l.c.m. if we include no more.

C. Here is one more example. Find the l.c.m. of 84 and 90.

$$84 = 2 \times 2 \times 3 \times 7$$

$$90 = 2 \times 3 \times 3 \times 5$$

Perhaps it will help to think of the problem in this way: What factors do I have to include beside

$$\underbrace{\overset{\uparrow}{2} \times \overset{\uparrow}{3} \times \overset{\uparrow}{3} \times \overset{\uparrow}{5}}_{90}$$

so that the expression will name a multiple of 84 ?

First we mark those numerals in the complete factorization of 84 that are already written in expressing 90.

$$\underbrace{\overset{\downarrow}{2} \times \overset{\downarrow}{3} \times 3 \times 5}_{90}$$

Then we add the remaining piece of the complete factorization of 84.

$$\overbrace{\underbrace{\overset{\uparrow}{2} \times \overset{\uparrow}{3} \times 3 \times 5}_{90} \times \overset{\uparrow}{2} \times \overset{\uparrow}{7}}^{84} = 1260 = 90 \times 14 = 84 \times 15$$

If we show the factors in order, we get

$$\overbrace{\underbrace{2 \times 2 \times 3 \times 3 \times 5}_{90} \times 7}^{84} = 1260$$

Imagine doing this problem the long way!

Use what we found in this example to compare

$$\frac{5}{84} \text{ and } \frac{7}{90}.$$

$\frac{5}{84} = \frac{5 \times 15}{84 \times 15} = \frac{75}{1260}$

$\frac{7}{90} = \frac{7 \times 14}{90 \times 14} = \frac{98}{1260}$

$\frac{5}{84} < \frac{7}{9}$

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Exercise Set 10

1. Find the least common multiple of each pair of numbers. Then show the set of all common multiples.

Example: 14 and 35

$$14 = 2 \times 7$$

$$35 = 5 \times 7$$

$$\text{l.c.m.} = \underbrace{2 \times \rlap{$\overbrace{}^{35}$}7}_{14} \times 5 = 70$$

Set of common multiples = {70, 140, 210 ...}.

a. 10 and 21 (210, 420, 630, ...)

b. 24 and 9 (72, 144, 216, ...)

c. 20 and 36 (180, 360, 540, ...)

d. 30 and 16 (240, 480, 720, ...)

2. Give an example of each: (Answers will vary)

a. A pair of numbers whose l.c.m. is their product. (4 and 5, 2 and 3, etc.)

b. A pair of numbers whose l.c.m. is one of the numbers (2 and 3 and 1

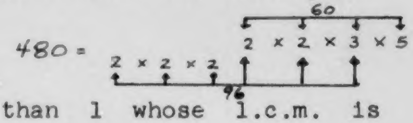
c. A pair of numbers for which neither a nor b is true. (4 and 6, 8 and 12)

3. A traffic light at one corner changes every 30 seconds. The traffic light at the next corner changes every 36 seconds. At a certain time they both change together. How long will it be until they change together again? (180 minutes)

$$\underbrace{2 \times \overbrace{2 \times 3 \times 5}^{30} \times 3}_{36} = 180$$

BRAINTWISTERS

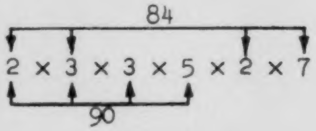
4. A "Discoverer" satellite goes directly over the north and south poles each time it circles the earth. It makes one circle in 96 minutes. It is directly over the north pole at noon. When will it next be over the north pole exactly on the hour? (8 hours)



5. a. Find two numbers greater than 1 whose l.c.m. is 96. (Hint: use the complete factorization of 96.) (3 and 32 and other pairs)

   b. The number 283 is not the l.c.m. of any pair of numbers except 1 and 283. What does this show about the factors of 283 ? (They are just 1 and 283.)

   c. Here is what we drew to help us find the l.c.m. of 84 and 90:



Form the product expression which uses the numerals with two arrows pointing to them. Can you find a meaning for such an expression? (2 × 3 is greatest common factor of 84 and 90)

LEAST COMMON DENOMINATOR

Objective: To rename fractions with least common denominators.

Vocabulary: Common denominator, least common denominator

Suggested Teaching Procedure

Teacher and pupils should work through material on "Least Common Denominator" together. Then pupils should work Exercise Sets 11 and 12 independently.

LEAST COMMON DENOMINATOR

Do you remember why we wanted to find common multiples? Let us use what we have learned to think again about how to compare $\frac{2}{3}$ and $\frac{7}{10}$.

We need to find a name for $\frac{2}{3}$ and a name for $\frac{7}{10}$ with the same denominator. We say that we want to find a common denominator for $\frac{2}{3}$ and $\frac{7}{10}$. We know that the common denominators for $\frac{2}{3}$ and $\frac{7}{10}$ are the common multiples of 3 and 10.

The least common multiple of 3 and 10 is the least common denominator of $\frac{2}{3}$ and $\frac{7}{10}$.

We can find the least common denominator for $\frac{2}{3}$ and $\frac{7}{10}$ in this way:

$$10 = 2 \times 5$$

3 is prime

l.c.m. of 10 and 3 is $\underbrace{2 \times 5}_{10} \times \overset{\overset{3}{\downarrow}}{3} = 30$.

To rename the fractions with the least common denominator we must also find n in $n \times 3 = 30$. Our diagram shows that

$$3 \times 10 = 30$$

$$10 \times 3 = 30.$$

Now we know

$$\frac{2}{3} = \frac{2 \times 10}{3 \times 10} = \frac{20}{30} \text{ and}$$

$$\frac{7}{10} = \frac{7 \times 3}{10 \times 3} = \frac{21}{30}.$$

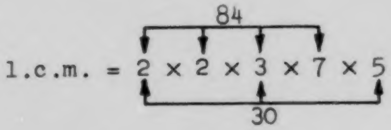
Notice that 60, 90, 120 ... are also common denominators.

We didn't really need our method with the small numbers 3 and 10. Here is an example with greater numbers. Which is greater $\frac{37}{84}$ or $\frac{13}{30}$?

First, we find the least common denominator.

$$84 = 2 \times 2 \times 3 \times 7$$

$$30 = 2 \times 3 \times 5$$



Now we want to express 84 and 30 as factors of the l.c.m. By looking at the arrows, we see that

$$\text{l.c.m.} = 84 \times 5 = 30 \times 14.$$

Now we can write:

$$\frac{37}{84} = \frac{37 \times 5}{84 \times 5} = \frac{185}{420}$$

$$\frac{13}{30} = \frac{13 \times 14}{30 \times 14} = \frac{182}{420}$$

We find, then, that

$$\frac{37}{84} > \frac{13}{30} .$$

Exercise Set 11

1. Rename each pair of numbers so the fractions have the least common denominator. Hint: Rename in simplest form before finding a least common denominator.

a. $\frac{1}{2}$ and $\frac{2}{4}$ ($\frac{1}{2}$ and $\frac{1}{2}$, or $\frac{2}{4}$ and $\frac{2}{4}$) d. $\frac{8}{5}$ and $\frac{8}{10}$ ($\frac{8}{5}$ and $\frac{4}{5}$)

b. $\frac{3}{5}$ and $\frac{3}{4}$ ($\frac{12}{20}$ and $\frac{15}{20}$) e. $\frac{4}{6}$ and $\frac{2}{3}$ ($\frac{2}{3}$ and $\frac{2}{3}$)

c. $\frac{6}{10}$ and $\frac{20}{25}$ ($\frac{3}{5}$ and $\frac{4}{5}$, or $\frac{30}{50}$ and $\frac{40}{50}$, etc) f. $\frac{6}{15}$ and $\frac{9}{14}$ ($\frac{28}{70}$ and $\frac{45}{70}$)

2. For each of the following pairs of fractions, find two other fractions which name the same two numbers and which have the least common denominator.

a. $\frac{2}{3}$ and $\frac{3}{4}$ ($\frac{8}{12}$ and $\frac{9}{12}$) e. $\frac{4}{5}$ and $\frac{7}{10}$ ($\frac{8}{10}$ and $\frac{7}{10}$)

b. $\frac{1}{4}$ and $\frac{2}{5}$ ($\frac{5}{20}$ and $\frac{8}{20}$) f. $\frac{7}{8}$ and $\frac{2}{3}$ ($\frac{21}{24}$ and $\frac{16}{24}$)

c. $\frac{5}{8}$ and $\frac{3}{5}$ ($\frac{25}{40}$ and $\frac{24}{40}$) g. $\frac{2}{5}$ and $\frac{1}{6}$ ($\frac{12}{30}$ and $\frac{5}{30}$)

d. $\frac{3}{8}$ and $\frac{5}{6}$ ($\frac{9}{24}$ and $\frac{20}{24}$) h. $\frac{1}{2}$ and $\frac{2}{3}$ ($\frac{3}{6}$ and $\frac{4}{6}$)

3. Which fraction names the greatest rational number?

a. $\frac{2}{3}$ or $\frac{3}{4}$ ($\frac{3}{4}$) e. $\frac{5}{7}$ or $\frac{2}{3}$ ($\frac{5}{7}$)

b. $\frac{3}{9}$ or $\frac{3}{5}$ ($\frac{3}{5}$) f. $\frac{5}{8}$ or $\frac{13}{16}$ ($\frac{13}{16}$)

c. $\frac{3}{4}$ or $\frac{4}{5}$ ($\frac{4}{5}$) g. $\frac{3}{5}$ or $\frac{2}{3}$ ($\frac{2}{3}$)

d. $\frac{4}{5}$ or $\frac{5}{6}$ ($\frac{5}{6}$) h. $\frac{5}{8}$ or $\frac{2}{3}$ ($\frac{2}{3}$)

4. Arrange in order from least to greatest.

$\frac{3}{4}$, $\frac{4}{5}$, and $\frac{7}{10}$ ($\frac{7}{10}$, $\frac{3}{4}$, $\frac{4}{5}$)

Exercise Set 12

1. Find which number is greater.

   a. $\frac{7}{18}$ or $\frac{61}{140}$ $\left(\frac{61}{140}\right)$

   b. $\frac{16}{27}$ or $\frac{13}{24}$ $\left(\frac{16}{27}\right)$

2. a. Which is longer, $\frac{1}{3}$ of a year or 121 days? ($\frac{1}{3}$ of a year is longer.)

   b. Which is longer, 2000 ft. or .3 of a mile? (2000 ft is longer.)

   (Hint: First find simplest fraction names.)

3. Which is greater?

   a. $\frac{19}{28}$ or $\frac{140}{210}$ $\left(\frac{19}{28}\right)$

   b. $\frac{45}{72}$ or $\frac{275}{400}$ $\left(\frac{275}{400}\right)$

4. List these in order of size from least to greatest.

   a. $\frac{19}{16}$, $\frac{55}{48}$, $\frac{43}{36}$ $\left(\frac{55}{48}, \frac{19}{16}, \frac{43}{36}\right)$

   b. 3 , $\frac{30}{9}$, $\frac{23}{8}$ $\left(\frac{23}{8}, 3, \frac{30}{9}\right)$

5. Roy is making a hammer toy for his little sister. He has wooden pegs $\frac{19}{32}$ in. in diameter. To make holes in the board he has two drills. One makes a hole $\frac{5}{8}$ in. in diameter. The other makes a hole $\frac{11}{16}$ in. in diameter. Will the pegs fit through both sizes of holes? (yes) Which drill should he use? ($\frac{5}{8}$ inch drill)

P303

6. On a number line, if you wanted to show both fourths and sixths you would mark the line in twelfths. How would you mark the line to show both of these?

   a. Tenths and sixths? (thirtieths)

   b. Sixteenths and thirds? (forty-eighths)

   c. Twelfths and ninths? (thirty-sixths)

7. Suppose that there will be either 4 people or 6 people at your party, counting yourself. Suppose also that you want to cut a cake before the party and want to divide the whole cake fairly among the people at the party. How would you cut it? (In twelfths)

SCALES ON NUMBER LINES

Objective: To develop ideas needed for picturing addition and subtraction of rational numbers on the number line.

Materials: Number lines scaled in ones, fourths, eighths, tenths, twelfths.

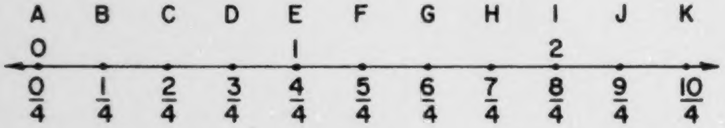
Vocabulary: Segment, measure of a segment, congruent segments, scale, scaled in halves, scaled in thirds, etc.

Suggested Teaching Procedure

In preparation for the following seven sections, the teacher should be thoroughly familiar with the development of addition and subtraction of whole numbers as presented in "Techniques of Addition and Subtraction", Grade 4, Chapters 3 and 6.

Work through the Exploration with the class.

Draw a number line scaled in fourths on the board:



What is the measure of these segments: $\overline{AB}$ $(\frac{1}{4})$; $\overline{AD}$ $(\frac{3}{4})$; $\overline{AH}$ $(\frac{7}{4})$; and so on.

One endpoint of a segment that has measure 1 is at $\frac{1}{4}$. What is the other endpoint? $(\frac{5}{4})$

Ask similar questions about points $\frac{2}{4}$ and $\frac{3}{4}$.

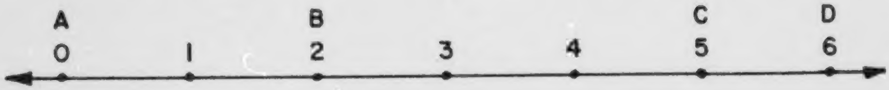
One endpoint of a segment that has measure 1 is at $\frac{6}{4}$. What is the other endpoint? (There are two answers: $\frac{2}{4}$ and $\frac{10}{4}$.)

Ask similar questions about other points.

Draw number lines scaled in sixths, eighths, tenths, and so on, as needed to help pupils understand the meaning of "scale" and review the meaning of "the measure of a segment."

SCALES ON NUMBER LINE

Exploration



On this number line, the marked points are equally spaced. Each point is labeled with a whole number. Some points are also named with letters.

The segment with endpoints at 0 and 1 is the unit segment.

1. Look at $\overline{AB}$. The number at point A is 0. The number at B is 2. $\overline{AB}$ is the union of how many segments, each congruent to the unit segment? (2)

2. What is the measure of $\overline{AC}$? (5)

3. What is the measure of $\overline{AD}$? (6)

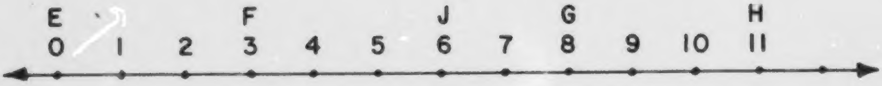
4. What is an easy way to tell the measure of any segment if one endpoint is at 0 ? (The number at the other endpoint is its measure.)

5. Look at $\overline{BC}$. B is the point labeled (2). C is the point labeled (5). $\overline{BC}$ is the union of (3) segments, each congruent to the unit segment. What is the measure of $\overline{BC}$? (3)

6. Name a segment whose measure is 4. ($\overline{BD}$)

7. Name a segment whose measure is 1. ($\overline{CD}$)

answers will vary



8. On the number line above name with letters: (Answers will vary. Some examples are:

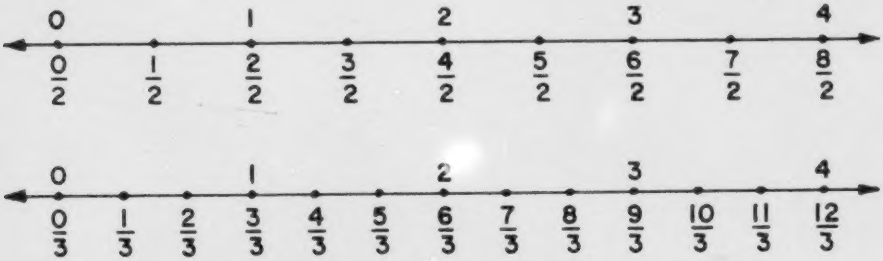
a. Three segments, measure of each is 3. ($\overline{EF}$, $\overline{FJ}$, $\overline{GH}$)

b. Two segments, measure of each is 5. ($\overline{JH}$, $\overline{FG}$)

c. One segment whose measure is 6. ($\overline{EJ}$)

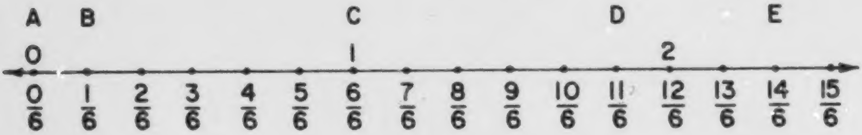
d. Two segments, measure of each is 8. ($\overline{FH}$, $\overline{EG}$)

As you know, we may separate a number line into congruent segments smaller than unit segments. Here are two examples:



The first line above uses a scale of halves. It is scaled in halves. The second line above uses a scale of thirds. It is scaled in thirds.

Generally we will show the scale we are using by the denominator of the fraction we use in marking the scale. If we are marking a scale in sixths, we will label a point $\frac{3}{6}$ rather than $\frac{6}{12}$, or $\frac{2}{4}$, or $\frac{1}{2}$. If we are marking a scale in fifths we will use the label $\frac{2}{5}$ rather than $\frac{4}{10}$, or $\frac{6}{15}$, or $\frac{8}{20}$.


A
B
C
D
E
0
1
2
$\frac{0}{6}$
$\frac{1}{6}$
$\frac{2}{6}$
$\frac{3}{6}$
$\frac{4}{6}$
$\frac{5}{6}$
$\frac{6}{6}$
$\frac{7}{6}$
$\frac{8}{6}$
$\frac{9}{6}$
$\frac{10}{6}$
$\frac{11}{6}$
$\frac{12}{6}$
$\frac{13}{6}$
$\frac{14}{6}$
$\frac{15}{6}$

9. a. This number line is scaled in (sixths).

Find the measures of these segments.

b. $\overline{AB}$ $(\frac{1}{6})$

c. $\overline{AC}$ $(\frac{6}{6})$

d. $\overline{AD}$ $(\frac{11}{6})$

e. $\overline{AE}$ $(\frac{14}{6})$

f. $\overline{BC}$ $(\frac{5}{6})$

g. $\overline{BD}$ $(\frac{10}{6})$

h. $\overline{BE}$ $(\frac{13}{6})$

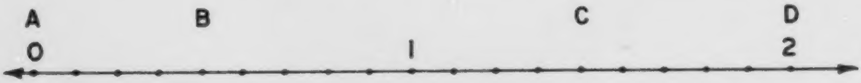
i. $\overline{CD}$ $(\frac{5}{6})$

j. $\overline{CE}$ $(\frac{8}{6})$

k. $\overline{DE}$ $(\frac{3}{6})$

Exercise Set 13

1. a. Look at this number line. Into how many congruent segments is the unit segment separated? (9)



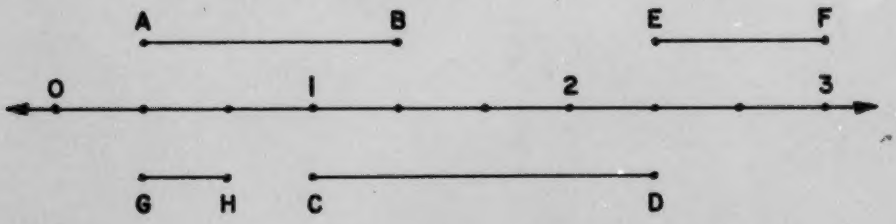
b. What is the scale? (ninths)

c. Write the set of fractions you would use to label the points from 0 to 2. $(\frac{0}{9}, \frac{1}{9}, \frac{2}{9}, \cdots \frac{18}{9})$

d. What number of your scale matches point A? $(\frac{0}{9})$ B? $(\frac{4}{9})$ C? $(\frac{13}{9})$ D? $(\frac{18}{9})$

e. What is the measure of $\overline{AB}$? $(\frac{4}{9})$ $\overline{AC}$? $(\frac{13}{9})$ $\overline{AD}$? $(\frac{18}{9})$ $\overline{BC}$? $(\frac{9}{9})$ $\overline{BD}$? $(\frac{14}{9})$ $\overline{CD}$? $(\frac{5}{9})$

2. On the number line below are pictured $\overline{AB}$, $\overline{CD}$, $\overline{EF}$ and $\overline{GH}$.



a. The points shown with dots could be labeled in a scale of (thirds).

b. What numbers match the points named with letters?

A. $\frac{1}{3}$ E. $\frac{7}{3}$
B. $\frac{4}{3}$ F. $\frac{9}{3}$
C. $\frac{3}{3}$ G. $\frac{1}{3}$
D. $\frac{7}{3}$ H. $\frac{2}{3}$

c. What is the measure of each segment?
($m\overline{AB} = 1$ or $\frac{3}{3}$; $m\overline{EF} = \frac{2}{3}$; $m\overline{GH} = \frac{1}{3}$; $m\overline{CD} = \frac{4}{3}$)

3. Do the three segments described in exercise a have the same measure? Answer the same question for exercises b and c.

a. Endpoints $\frac{6}{3}$ and $\frac{9}{3}$, $\frac{4}{2}$ and $\frac{6}{2}$, $\frac{2}{8}$ and $\frac{10}{8}$ (yes)

b. Endpoints $\frac{5}{4}$ and $\frac{11}{4}$, $\frac{3}{2}$ and $\frac{6}{2}$, $\frac{1}{3}$ and $\frac{5}{3}$ (no)

c. Endpoints $\frac{4}{12}$ and $\frac{10}{12}$, $\frac{0}{2}$ and $\frac{1}{2}$, $\frac{3}{8}$ and $\frac{7}{8}$ (yes)

4. Rename each of these numbers with a fraction that could be used to label a point of a scale of twenty-fourths.

Example: $\frac{1}{2} = \frac{12}{24}$

a. $\frac{1}{3}$ $(\frac{8}{24})$ b. $\frac{2}{4}$ $(\frac{12}{24})$ c. $\frac{2}{6}$ $(\frac{8}{24})$ d. $\frac{11}{12}$ $(\frac{22}{24})$ e. $\frac{5}{8}$ $(\frac{15}{24})$

5. Which of these numbers cannot be renamed by a fraction which could be used to mark a scale in eighteenths? (b, c, e, g)

a. $\frac{1}{6}$ d. $\frac{2}{9}$ f. $\frac{2}{3}$

b. $\frac{5}{12}$ e. $\frac{3}{4}$ g. $\frac{1}{10}$

c. $\frac{7}{16}$

ADDITION OF RATIONAL NUMBERS ON THE NUMBER LINE

Objectives: To develop the meaning of addition of rational numbers by considering the measures of segments on the number line, and the measure of their union when the segments are placed "end-to-end".

To develop a procedure for computing the sum of two rational numbers whose measures are in the same scale--that is, when the measures are named by fractions which have the same denominator.

To show that addition of rational numbers named by fractions with the same denominator reduces to addition of whole numbers.

Vocabulary: Sum, addend, addition

Materials: Number lines, dittoed number lines for children. (See "Materials")

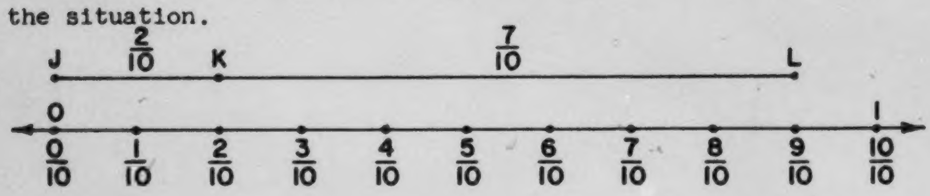
Suggested Teaching Procedure

|| Write this problem on the board: ||

The towns of Justin, Karver and Lind are on a straight road. Karver is between Justin and Lind. It is $\frac{2}{10}$ mi. from Justin to Karver and $\frac{7}{10}$ mi. from Karver to Lind. How far is it from Justin to Lind?

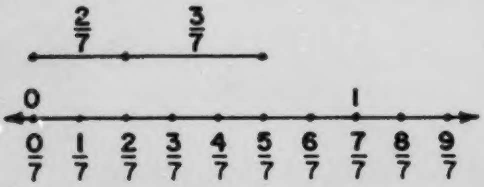
Do you know the answer to this problem? Can you think of any way to picture it? (With a number line) How do we scale the number line? (In tenths) Why? (The measures of the sections of the road are expressed in tenths.)

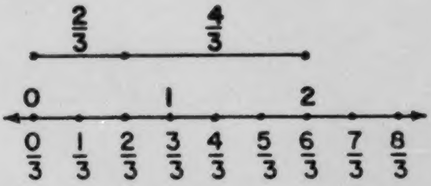
Here is a number line diagram. Imagine it as a picture of the situation.



What does $\overline{JK}$ represent? (The section of the road between Justin and Karver.) What does $\overline{KL}$ represent? (The section of the road between Karver and Lind.) What does $\overline{JL}$ represent? (The section of the road between Justin and Lind.) How would you ordinarily express the distance from Justin to Lind? ($\frac{9}{10}$)

Here are some number line diagrams. They show how to put copies of segments together "end-to-end", so their union is a segment. For each diagram tell the measures of both segments and the measure of their union.


2/7
3/7
0
1
0/7
1/7
2/7
3/7
4/7
5/7
6/7
7/7
8/7
9/7


2/3
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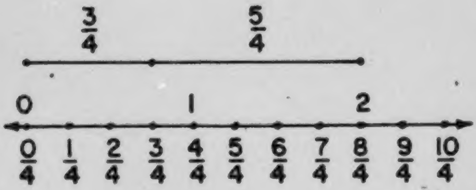
6/3
7/3
8/3


3/4
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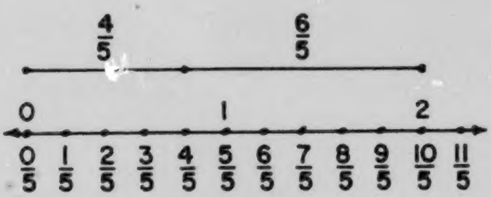
9/4
10/4


4/5
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9/5
10/5
11/5

The method we have used for building a new line segment out of two others is a way we used earlier of picturing addition of whole numbers. Because the same method also pictures an operation on rational numbers, we simply agree to call this operation addition. In this language we can express a relation between measures of length by using the usual symbol (+) to indicate addition. In our examples the relations we have found are:

$$\frac{2}{7} + \frac{3}{7} = \frac{5}{7} \qquad \frac{3}{4} + \frac{5}{4} = \frac{8}{4}$$

$$\frac{2}{3} + \frac{4}{3} = \frac{6}{3} \qquad \frac{4}{5} + \frac{6}{5} = \frac{10}{5}$$

How is the number line scaled to show: $\frac{2}{7} + \frac{3}{7}$? (in sevenths); $\frac{3}{4} + \frac{5}{4}$? (in fourths); $\frac{2}{3} + \frac{4}{3}$? (in thirds); $\frac{4}{5} + \frac{6}{5}$? (in fifths).

We really don't need diagrams to compute sums like $\frac{3}{4} + \frac{2}{4}$, $\frac{5}{8} + \frac{4}{8}$, or $\frac{7}{3} + \frac{6}{3}$. For $\frac{3}{4} + \frac{2}{4}$ we can count by fourths from $\frac{3}{4}$. We think 3 fourths, 4 fourths, 5 fourths. For $\frac{5}{8} + \frac{4}{8}$, how do we count? (We count by eighths from $\frac{5}{8}$. We think, "5 eighths 6 eighths, 7 eighths, 8 eighths, 9 eighths.")

This counting really amounts to addition of two whole numbers. We think

$$\frac{3}{4} + \frac{2}{4} = \frac{3 + 2}{4}, \qquad \frac{5}{8} + \frac{4}{8} \quad \frac{5 + 4}{8}$$

Is there any way to get the results for these additions without using the number line or counting? (Yes. The numerator of the result seems to be the sum of the numerators of the addends. The denominator of the result seems to be the common denominator of the addends.)

Are these mathematical sentences true? Use number lines to decide.

a. $\frac{1}{2} + \frac{1}{2} = \frac{1 + 1}{2}$ c. $\frac{1}{3} + \frac{1}{3} = \frac{1 + 1}{3}$

b. $\frac{3}{8} + \frac{2}{8} = \frac{3 + 2}{8}$ d. $\frac{1}{6} + \frac{3}{6} = \frac{1 + 3}{6}$

What is the simplest fraction name for $\frac{1 + 1}{2}$? ($\frac{2}{2}$) for $\frac{1 + 1}{3}$? ($\frac{2}{3}$) for $\frac{3 + 2}{8}$? ($\frac{5}{8}$) for $\frac{1 + 3}{6}$? ($\frac{4}{6}$)

In computing, the following form illustrating $\frac{3}{4} + \frac{4}{4}$ should be used: $\frac{3}{4} + \frac{4}{4} = \frac{3 + 4}{4} = \frac{7}{4}$.

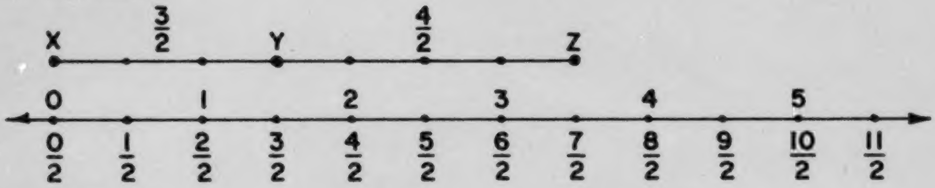
In this and succeeding sections, answers to exercises are often rational numbers. It is suggested that any appropriate name for that number be accepted unless there are directions specifying a particular kind of name. Thus $\frac{4}{3}$, $\frac{8}{6}$, and later $1\frac{1}{3}$ would all be accepted. Of course, you may wish to point out advantages of particular kinds of names in particular circumstances, and state that, in the absence of any special considerations, the fraction with the smallest possible denominator is the customarily preferred name.

ADDITION OF RATIONAL NUMBERS ON THE NUMBER LINE

Exploration

Building a new line segment out of two others is a way of picturing addition of whole numbers. Because the same method also pictures an operation on rational numbers, we agree to call this operation addition. We express a relation between measures of length using the usual symbol "+" to indicate addition.

The addition, $\frac{3}{2} + \frac{4}{2}$ may be shown on a number line scaled in halves.



$\overline{XY}$ is the union of three congruent segments, each with measure $\frac{1}{2}$, so the measure of $\overline{XY}$ is $\frac{3}{2}$.

$\overline{YZ}$ is the union of four congruent segments, each with measure $\frac{1}{2}$, so the measure of $\overline{YZ}$ is $\frac{4}{2}$.

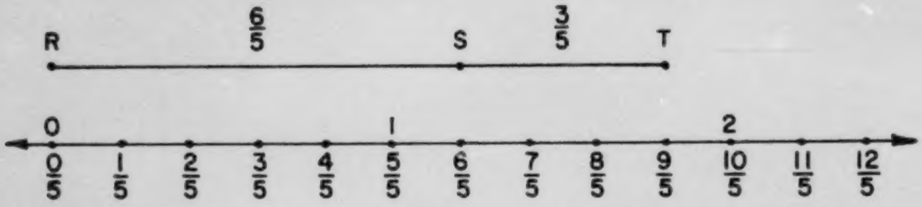
$\overline{XZ}$ is the union of $(3 + 4)$, or 7 congruent segments, each with measure $\frac{1}{2}$. The measure of $\overline{XZ}$ is $\frac{7}{2}$.

We write

$$\frac{3}{2} + \frac{4}{2} = \frac{3 + 4}{2} = \frac{7}{2} .$$

We can think of $\frac{7}{2}$ as another name for $\frac{3}{2} + \frac{4}{2}$.

The addition $\frac{6}{5} + \frac{3}{5}$ may be shown on a number line scaled in fifths.


R
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S
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2
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$\frac{3}{5}$
$\frac{4}{5}$
$\frac{5}{5}$
$\frac{6}{5}$
$\frac{7}{5}$
$\frac{8}{5}$
$\frac{9}{5}$
$\frac{10}{5}$
$\frac{11}{5}$
$\frac{12}{5}$

1. $\overline{RS}$ is the union of (6) congruent segments, each with measure $(\frac{1}{5})$. The measure of $\overline{RS}$ is $(\frac{6}{5})$.

2. $\overline{ST}$ is the union of (3) congruent segments, each with measure $(\frac{1}{5})$. The measure of $\overline{ST}$ is $(\frac{3}{5})$.

3. $\overline{RT}$ is the union of (9) congruent segments, each with measure $(\frac{1}{5})$. The measure of $\overline{RT}$ is $\frac{(6+3)}{5}$ or $(\frac{9}{5})$.

We write

$$\frac{6}{5} + \frac{3}{5} = \frac{6+3}{5} = \frac{9}{5}$$

We can think of $\frac{9}{5}$ as another name for $\frac{6}{5} + \frac{3}{5}$.

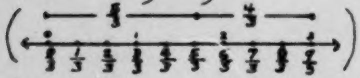
Exercise Set 14

1. Use number line diagrams to show each of these sums.

a. $\frac{5}{3} + \frac{4}{3}$

b. $\frac{2}{4} + \frac{7}{4}$

c. $\frac{3}{2} + \frac{4}{2}$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0
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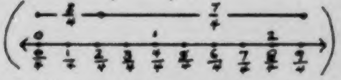
$\frac{0}{3}$ $\frac{1}{3}$ $\frac{2}{3}$ $\frac{3}{3}$ $\frac{4}{3}$ $\frac{5}{3}$ $\frac{6}{3}$ $\frac{7}{3}$ $\frac{8}{3}$ $\frac{9}{3}$
)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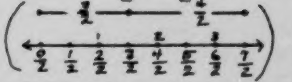
$\frac{0}{4}$ $\frac{1}{4}$ $\frac{2}{4}$ $\frac{3}{4}$ $\frac{4}{4}$ $\frac{5}{4}$ $\frac{6}{4}$ $\frac{7}{4}$ $\frac{8}{4}$ $\frac{9}{4}$
)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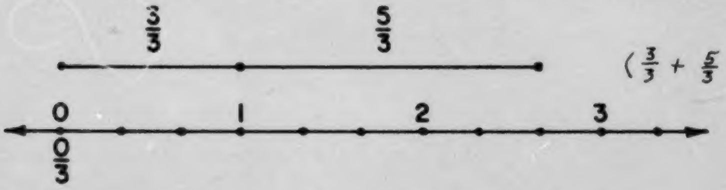
$\frac{0}{2}$ $\frac{1}{2}$ $\frac{2}{2}$ $\frac{3}{2}$ $\frac{4}{2}$ $\frac{5}{2}$ $\frac{6}{2}$ $\frac{7}{2}$
)

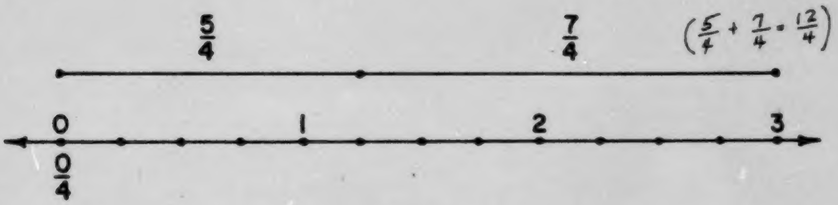
2. Write a mathematical sentence for each of the following diagrams:

a.


$\frac{3}{3}$
$\frac{5}{3}$
0
1
2
3
$\frac{0}{3}$

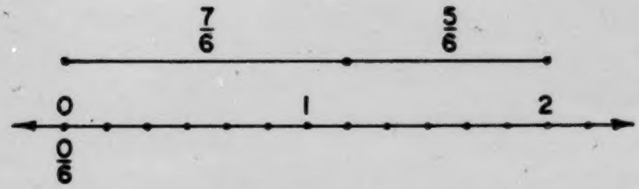
($\frac{3}{3} + \frac{5}{3} = \frac{8}{3}$)

b.


$\frac{5}{4}$
$\frac{7}{4}$
0
1
2
3
$\frac{0}{4}$

($\frac{5}{4} + \frac{7}{4} = \frac{12}{4}$)

c.


$\frac{7}{6}$
$\frac{5}{6}$
0
1
2
$\frac{0}{6}$

($\frac{7}{6} + \frac{5}{6} = \frac{12}{6}$)

3. Copy and find a fraction name for each of these sums.

Exercise a is done for you.

a. $\frac{3}{4} + \frac{5}{4} = \frac{3+5}{4} = \frac{8}{4}$

b. $\frac{5}{6} + \frac{9}{6}$ ($\frac{14}{6}$)

c. $\frac{4}{5} + \frac{7}{5}$ ($\frac{11}{5}$)

d. $\frac{7}{8} + \frac{9}{8}$ ($\frac{16}{8}$)

e. $\frac{5}{12} + \frac{9}{12}$ ($\frac{14}{12}$)

4. Find a fraction name for n, if:

a. $n = \frac{8}{3} + \frac{7}{3}$ $(\frac{15}{3})$

b. $n = \frac{5}{8} + \frac{7}{8}$ $(\frac{12}{8})$

c. $n = \frac{3}{4} + \frac{7}{4}$ $(\frac{10}{4})$

d. $n = \frac{5}{6} + \frac{3}{6}$ $(\frac{8}{6})$

5. Copy each of the following and represent the sum in simplest form. Exercise a is done for you

a. $\frac{1}{4} + \frac{1}{4} = \frac{1 + 1}{4} = \frac{2}{4} = \frac{1}{2}$

b. $\frac{2}{8} + \frac{4}{8}$ $(\frac{3}{4})$

c. $\frac{1}{6} + \frac{3}{6}$ $(\frac{2}{3})$

d. $\frac{7}{10} + \frac{1}{10}$ $(\frac{4}{5})$

e. $\frac{3}{8} + \frac{1}{8}$ $(\frac{1}{2})$

f. $\frac{1}{6} + \frac{1}{6}$ $(\frac{1}{3})$

g. $\frac{3}{12} + \frac{1}{12}$ $(\frac{1}{3})$

6. BRAINTWISTERS

a. $\frac{5}{6}$ is the result of adding two rational numbers. The fraction name for each addend has a denominator 6. What are two possible addends? $(\frac{1}{6} + \frac{4}{6}$ or $\frac{2}{6} + \frac{3}{6})$

b. $\frac{5}{6}$ is the result of adding two rational numbers. Each fraction name has a denominator 12. What are two possible addends? $(\frac{6}{12} + \frac{4}{12}$ or $\frac{9}{12} + \frac{1}{12}$, etc.)

c. $\frac{5}{6}$ is the result of adding two rational numbers. One fraction has a denominator of 4 and the other fraction has a denominator 12. What are two possible addends? $(\frac{3}{4} + \frac{1}{12}; \frac{2}{4} + \frac{4}{12}$, etc.)

d. $\frac{7}{12}$ is the result of adding two rational numbers. One fraction has a denominator of 3 and the other has a denominator of 4. What are two possible addends? $(\frac{1}{4} + \frac{1}{3})$

SUBTRACTION OF RATIONAL NUMBERS

Objectives: To show that subtraction of rational numbers, like subtraction of whole numbers, is a process of finding one addend when the other addend and sum are known.

To picture subtraction of rational numbers with segments on the number line.

To show that subtraction of rational numbers named by fractions with the same denominator reduces to subtraction of whole numbers.

Vocabulary: Sum, addend, unknown addend.

Materials: Number lines, dittoed number lines for children.

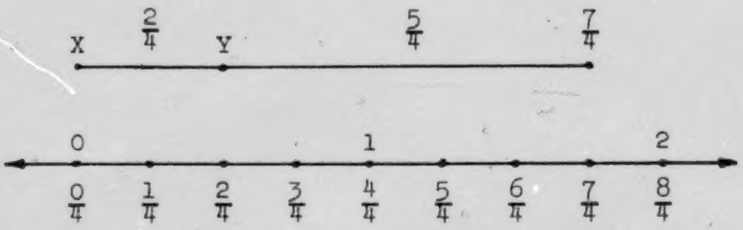
Suggested Teaching Procedure

> If your pupils are not accustomed to thinking of subtraction of whole numbers as finding an addend, you may wish to proceed as follows:

In the mathematical sentence $\frac{2}{3} + \frac{1}{3} = n$, what are the addends? ($\frac{2}{3}$ and $\frac{1}{3}$) What is the sum? (n) In the mathematical sentence $p = \frac{6}{4} - \frac{2}{4}$, what is the sum? ($\frac{6}{4}$) What are the addends? p and $\frac{2}{4}$)

> It may be necessary to write mathematical sentences with whole numbers to help pupils learn this vocabulary.

Draw a number line, as shown, on the board.



Does the number line show $\frac{2}{4} + \frac{5}{4} = \frac{7}{4}$? (Yes) What scale is used in the diagram? (Fourths) What do we call $\frac{2}{4}$ and $\frac{5}{4}$? (Addends) What do we call $\frac{7}{4}$? (Sum)

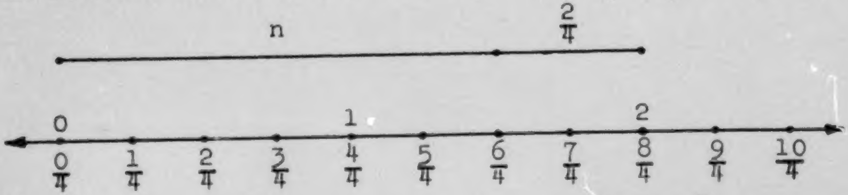
Does the number line show $\frac{7}{4} - \frac{5}{4} = \frac{2}{4}$? (Yes) In $\frac{7}{4} - \frac{5}{4} = \frac{2}{4}$, what are the addends? ($\frac{5}{4}$ and $\frac{2}{4}$) What is the sum? ($\frac{7}{4}$) Does the sentence $\frac{2}{4} + \frac{5}{4} = \frac{7}{4}$ express the same relationship among $\frac{2}{4}$, $\frac{5}{4}$, and $\frac{7}{4}$ as does $\frac{7}{4} - \frac{5}{4} = \frac{2}{4}$? (Yes)

Write the mathematical sentence $\frac{1}{2} + \frac{7}{2} = \frac{8}{2}$, using the sign for subtraction. ($\frac{8}{2} - \frac{1}{2} = \frac{7}{2}$ or $\frac{8}{2} - \frac{7}{2} = \frac{1}{2}$)

|| Use other examples as needed by your class ||

In $\frac{5}{3} + \frac{4}{3} = n$, n is the sum. Do the sentences $n = \frac{5}{3} - \frac{4}{3}$ and $n - \frac{4}{3} = \frac{5}{3}$ state the same relationship? (Yes)

Look at the number line. What scale is used? (Fourths) What mathematical sentence is shown? ($n + \frac{2}{4} = \frac{8}{4}$, or $\frac{8}{4} - \frac{2}{4} = n$)



What is one addend? What is the other addend? ($\frac{2}{4}$ and n) What is the sum? ($\frac{8}{4}$) Look at the number line. What number is n? ($\frac{6}{4}$) How could we find a name for n without using the number line?

$$\left(n = \frac{8}{4} - \frac{2}{4};\ \text{therefore,}\ n = \frac{8 - 2}{4} = \frac{6}{4}\right)$$

Emphasize the relationship between addition and subtraction and the fact that there is similarity in the form for obtaining results.

Furnish more examples like the one above and ask similar questions.

There are no new techniques needed for computing addends that are rational numbers. If the sum and one addend are named by fractions in the same scale, the subtraction is reduced to subtraction of whole numbers.

In computing, the following form, illustrated for $\frac{10}{8} - \frac{4}{8}$ should be used.

$$\frac{10}{8} - \frac{4}{8} = \frac{10 - 4}{8} = \frac{6}{8}$$

SUBTRACTION OF RATIONAL NUMBERS

Addition and subtraction are operations on two numbers. The result of each operation is a single number.

The result of adding $\frac{2}{3}$ and $\frac{5}{3}$ is $\frac{7}{3}$. We have added. We call $\frac{2}{3}$ and $\frac{5}{3}$ addends. We call $\frac{7}{3}$ the sum.

Addition of rational numbers may be expressed with fraction numerals as shown on the right. In addition, two addends are known. We wish to find the sum.

$$\frac{2}{3} + \frac{5}{3} = \frac{7}{3} \qquad \begin{array}{rl} \frac{2}{3} & \text{addend} \\ + \frac{5}{3} & \text{addend} \\ \hline \frac{7}{3} & \text{sum} \end{array}$$

When we think about $\frac{7}{3}$ and $\frac{5}{3}$ and get a result of $\frac{2}{3}$, we have subtracted. We call $\frac{5}{3}$ and $\frac{2}{3}$ addends. We call $\frac{7}{3}$ the sum.

Subtraction of rational numbers may be expressed with fraction numerals as shown at the right. In subtraction, the sum and one addend are known. We wish to find the other addend.

$$\frac{7}{3} - \frac{5}{3} = \frac{2}{3} \qquad \begin{array}{rl} \frac{7}{3} & \text{sum} \\ - \frac{5}{3} & \text{addend} \\ \hline \frac{2}{3} & \text{addend} \end{array}$$

The mathematical sentences

$$\frac{2}{3} + \frac{5}{3} = \frac{7}{3}, \quad \frac{7}{3} - \frac{5}{3} = \frac{2}{3}, \quad \text{and} \quad \frac{7}{3} - \frac{2}{3} = \frac{5}{3}$$

express the same relationship among $\frac{2}{3}$, $\frac{5}{3}$, and $\frac{7}{3}$.

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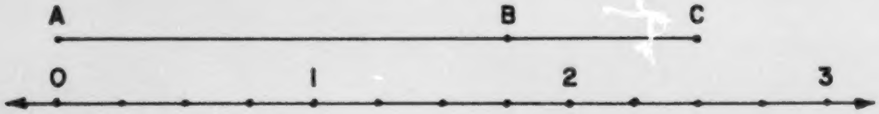
Look at the diagram below. Use the measures of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ to write three mathematical sentences which express the same relationship.



$$\left(\frac{7}{4} + \frac{3}{4} = \frac{10}{4}\right)$$

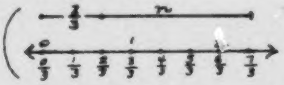
$$\left(\frac{10}{4} - \frac{7}{4} = \frac{3}{4}\right)$$

$$\left(\frac{10}{4} - \frac{3}{4} = \frac{7}{4}\right)$$

Exercise Set 15

1. Use number line diagrams to picture these relationships. What number does n represent?

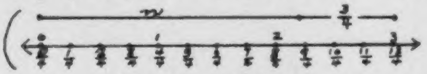
a. $\frac{7}{3} - \frac{2}{3} = n$



$\frac{2}{3}$
n
0
1
2
$\frac{0}{3}$ $\frac{1}{3}$ $\frac{2}{3}$ $\frac{3}{3}$ $\frac{4}{3}$ $\frac{5}{3}$ $\frac{6}{3}$ $\frac{7}{3}$

($n = \frac{5}{3}$)

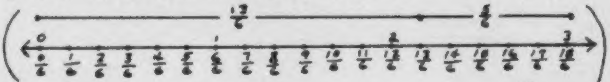
b. $n + \frac{3}{4} = \frac{12}{4}$



n
$\frac{3}{4}$
0
1
2
3
$\frac{0}{4}$ $\frac{1}{4}$ $\frac{2}{4}$ $\frac{3}{4}$ $\frac{4}{4}$ $\frac{5}{4}$ $\frac{6}{4}$ $\frac{7}{4}$ $\frac{8}{4}$ $\frac{9}{4}$ $\frac{10}{4}$ $\frac{11}{4}$ $\frac{12}{4}$

($n = \frac{9}{4}$)

c. $\frac{13}{6} + \frac{5}{6} = n$

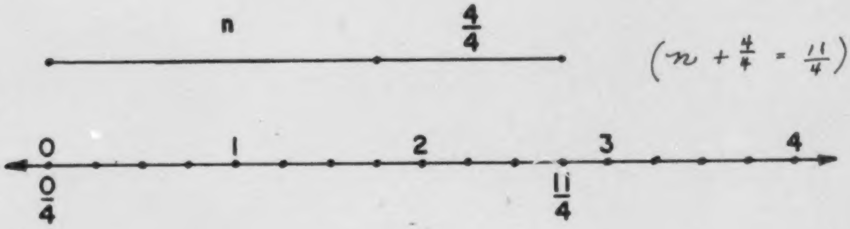


$\frac{13}{6}$
$\frac{5}{6}$
0
1
2
3
$\frac{0}{6}$ $\frac{1}{6}$ $\frac{2}{6}$ $\frac{3}{6}$ $\frac{4}{6}$ $\frac{5}{6}$ $\frac{6}{6}$ $\frac{7}{6}$ $\frac{8}{6}$ $\frac{9}{6}$ $\frac{10}{6}$ $\frac{11}{6}$ $\frac{12}{6}$ $\frac{13}{6}$ $\frac{14}{6}$ $\frac{15}{6}$ $\frac{16}{6}$ $\frac{17}{6}$ $\frac{18}{6}$

($n = \frac{18}{6}$)

2. What mathematical sentences are pictured in the diagrams below?

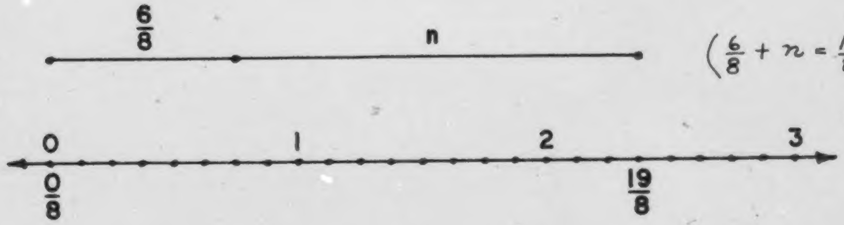
a.



n
$\frac{4}{4}$
0
1
2
3
4
$\frac{0}{4}$
$\frac{11}{4}$

($n + \frac{4}{4} = \frac{11}{4}$)

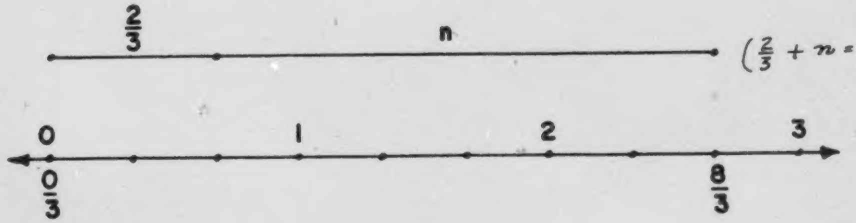
b.



$\frac{6}{8}$
n
0
1
2
3
$\frac{0}{8}$
$\frac{19}{8}$

($\frac{6}{8} + n = \frac{19}{8}$)

c.



$\frac{2}{3}$
n
0
1
2
3
$\frac{0}{3}$
$\frac{8}{3}$

($\frac{2}{3} + n = \frac{8}{3}$)

3. Copy each sentence and find a fraction name for n.

Exercise a is done for you.

a. $n = \frac{5}{3} - \frac{1}{3} = \frac{5 - 1}{3} = \frac{4}{3}$

b. $\frac{7}{5} - \frac{3}{5} = n$ $\left(\frac{4}{5}\right)$

c. $\frac{17}{8} - \frac{9}{8} = n$ $\left(\frac{8}{8}\right)$

d. $n = \frac{14}{12} - \frac{9}{12}$ $\left(\frac{5}{12}\right)$

e. $n = \frac{15}{16} - \frac{8}{16}$ $\left(\frac{7}{16}\right)$

f. $\frac{8}{4} - \frac{5}{4} = n$ $\left(\frac{3}{4}\right)$

g. $\frac{11}{5} - \frac{7}{5} = n$ $\left(\frac{4}{5}\right)$

4. Copy each sentence and find the other addend. Name each answer in simplest fraction form.

a. $\frac{5}{6} - \frac{2}{6} = n$ $\left(\frac{1}{2}\right)$

b. $\frac{7}{8} - \frac{5}{8} = n$ $\left(\frac{1}{4}\right)$

c. $\frac{7}{4} - \frac{5}{4} = n$ $\left(\frac{1}{2}\right)$

d. $\frac{8}{10} - \frac{4}{10} = n$ $\left(\frac{2}{5}\right)$

e. $\frac{9}{16} - \frac{5}{16} = n$ $\left(\frac{1}{4}\right)$

f. $\frac{7}{6} - \frac{5}{6} = n$ $\left(\frac{1}{3}\right)$

PICTURING ADDITION AND SUBTRACTION WITH REGIONS

Objective: To show that addition and subtraction of rational numbers can be pictured with regions, as well as segments.

Vocabulary: Unit region.

Materials: Models of rectangular regions and circular regions

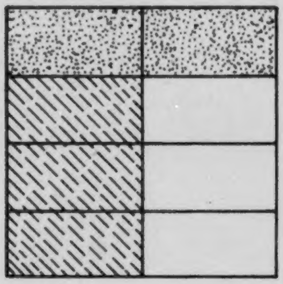
Suggested Teaching Procedure

> Work through the Exploration with the class. Use additional models as needed. Emphasize that a relationship between two addends and a sum may be stated by different mathematical sentences.

PICTURING ADDITION AND SUBTRACTION WITH REGIONS

Exploration

You have seen how number lines can be used to picture addition and subtraction of rational numbers. Regions can be used also.



A

1. Figure A represents a unit region. Each of the small regions is $\underline{(\frac{1}{8})}$ of the unit region.

2. The dotted region is $\underline{(\frac{2}{8})}$ of the unit region.

3. The shaded region is $\underline{(\frac{3}{8})}$ of the unit region.

4. The unshaded region is $\underline{(\frac{3}{8})}$ of the unit region.

5. Which regions picture the mathematical sentence

$$\frac{3}{8} + \frac{2}{8} = \frac{5}{8} \,?$$

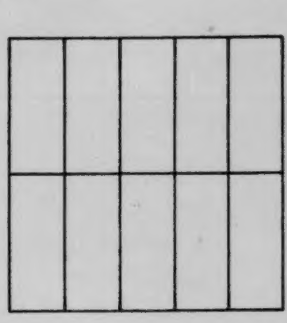
(The shaded and dotted regions or
the unshaded and dotted regions)

6. Write two other mathematical sentences which express the same relationship among $\frac{3}{8}$, $\frac{2}{8}$, and $\frac{5}{8}$. $\left(\begin{array}{l} \frac{5}{8} - \frac{2}{8} = \frac{3}{8} \\ \frac{5}{8} - \frac{3}{8} = \frac{2}{8} \end{array}\right)$

7. Write a mathematical sentence pictured by the dotted and unshaded regions. $\left(\frac{2}{8} + \frac{3}{8} = \frac{5}{8}, \text{ etc.}\right)$

8. Write three mathematical sentences suggested by the unit region and the dotted regions. $\left(\begin{array}{l} \frac{8}{8} = \frac{2}{8} + n; \quad \frac{8}{8} - \frac{2}{8} = n; \\ \frac{8}{8} - n = \frac{2}{8} \end{array}\right)$

9. The unit region and the unshaded region suggest that $\frac{3}{8} + \underline{\left(\frac{5}{8}\right)} = \frac{8}{8}$. Write two other mathematical sentences for this relationship. $\left(\frac{8}{8} - \frac{3}{8} = \frac{5}{8}; \quad \frac{8}{8} - \frac{5}{8} = \frac{3}{8}\right)$



B

10. Trace figure B shown above. Shade some parts and write three mathematical sentences for your picture.

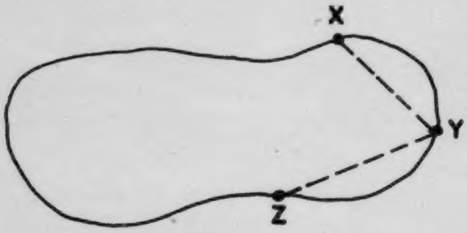
(answers will vary)

Exercise Set 16

Write a mathematical sentence for each problem. Use a number line if you need help in writing the mathematical sentence or finding the answer. Use simplest names for rational numbers used in your answers.

1. Susan needs $\frac{2}{3}$ yard ribbon to wrap one present and $\frac{2}{3}$ yard of ribbon for another. How much does she need?
   ($\frac{2}{3} + \frac{2}{3} = n$. She needed $\frac{4}{3}$ yards ribbon.)

2. Below is a map of a lake. Three friends live at the points marked X, Y, and Z. X and Y are $\frac{3}{10}$ miles apart. Y and Z are $\frac{2}{10}$ miles apart. How long a boat trip is it from X to Z by way of Y ? ($\frac{3}{10} + \frac{2}{10} = n$. The boat trip is $\frac{1}{2}$ mile long.)



3. There were 12 chapters in Mary's book. One day she read 2 chapters. The following day she read 1 chapter. What rational number best describes the part of the book she read on the two days? ($\frac{2}{12} + \frac{1}{12} = n$. She read $\frac{1}{4}$ of the book on the two days.)

SCALES FOR PICTURING ADDITION

Objectives: To show that to picture addition of rational numbers named by fractions with different denominators, it is necessary to find a suitable scale.

To show that a suitable scale is one in which the number of congruent segments in each unit segment is a common multiple of the denominators of the original fraction names.

To review renaming two rational numbers by fractions using the lowest common denominator.

Materials: Number lines, dittoed number lines for children

Suggested Teaching Procedure

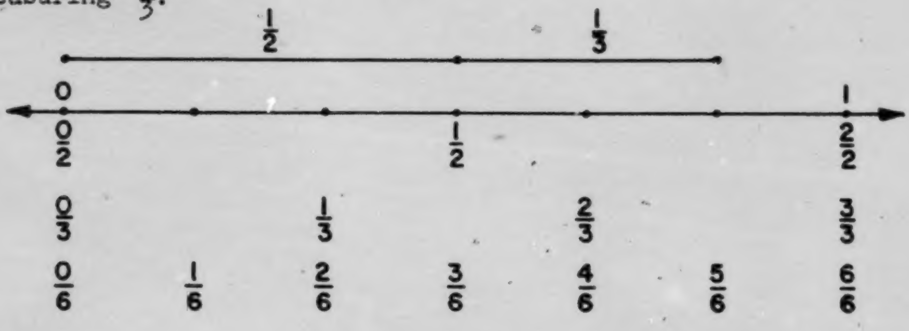
Addition of rational numbers has been related to finding a measure for the union of two segments placed "end-to-end" on the number line. In that case, the numbers were named by fractions with the same denominator, so the selection of a suitable scale presented no problem. The discussion in this section deals with the question of finding a suitable scale for picturing the sum of two numbers named by fractions with different denominators. It thus provides a geometric picture of the process of finding a common denominator and of renaming the numbers to be added.

It would be well to build up the first diagram in the text in stages, using the chalk board. Show first the scale for halves, then introduce the scale for thirds on the same line, and ask the pupils to think of a scale which will provide fraction labels for the points of both scales.

Work through the rest of the section with the pupils.

SCALES FOR PICTURING ADDITION

To picture $\frac{1}{2} + \frac{1}{3}$ on a number line, we need a suitable scale. A scale of sixths can show segments measuring $\frac{1}{2}$ and segments measuring $\frac{1}{3}$.

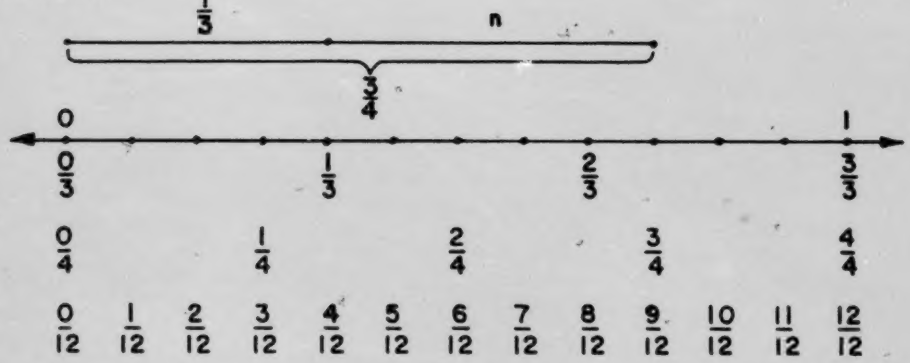


1/2
1/3
0
1
0/2
1/2
2/2
0/3
1/3
2/3
3/3
0/6
1/6
2/6
3/6
4/6
5/6
6/6

This diagram suggests that $\frac{1}{2} + \frac{1}{3}$ can be written as

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6}. \qquad \text{Now} \quad \frac{3}{6} + \frac{2}{6} = \frac{3+2}{6} = \frac{5}{6}.$$

To show $\frac{3}{4} - \frac{1}{3}$ on a number line we may use a scale of twelfths.



1/3
n
3/4
0
1
0/3
1/3
2/3
3/3
0/4
1/4
2/4
3/4
4/4
0/12
1/12
2/12
3/12
4/12
5/12
6/12
7/12
8/12
9/12
10/12
11/12
12/12

We think, $n = \frac{3}{4} - \frac{1}{3}$; so $n + \frac{1}{3} = \frac{3}{4}$. The number line shows that $n = \frac{5}{12}$. This suggests that $\frac{3}{4} - \frac{1}{3}$ can be written as

$$\frac{3}{4} - \frac{1}{3} = \frac{9}{12} - \frac{4}{12}. \qquad \text{Now,} \quad \frac{9}{12} - \frac{4}{12} = \frac{9-4}{12} = \frac{5}{12}.$$

Can we always find a suitable scale to picture the sum of rational numbers? Consider $\frac{3}{8} + \frac{1}{6}$.

To find a suitable scale we must find a scale such that each $\frac{1}{8}$ segment and each $\frac{1}{6}$ segment is separated into a whole number of smaller segments. For eighths, we can separate each $\frac{1}{8}$ segment into

2 congruent segments. Then there will be 2×8 parts in each unit segment.

3 congruent segments. Then there will be 3×8 parts in each unit segment.

4 congruent segments. Then there will be 4×8 parts in each unit segment.

Subdividing each $\frac{1}{8}$ segment in these ways suggests scales in which the denominators of the fractions are the set of multiples of 8. $\{8, 16, 24, 32, 40, 48 \ldots\}$ Subdividing sixths suggests the set of multiples of 6. $\{6, 12, 18, 24, 30, 36, \ldots\}$ A scale which can be used to picture addition of eighths and sixths will be one in which the denominator of the fraction is a common multiple of 8 and 6. The easiest scale to use is the one in which the l.c.m. of 8 and 6 is used, that is, the least common denominator for $\frac{3}{8}$ and $\frac{5}{6}$.

You know how to find the l.c.d. of $\frac{3}{8}$ and $\frac{5}{6}$. It is the l.c.m. of 8 and 6.

$$8 = 2 \times 2 \times 2 \qquad\qquad 6 = 2 \times 3$$

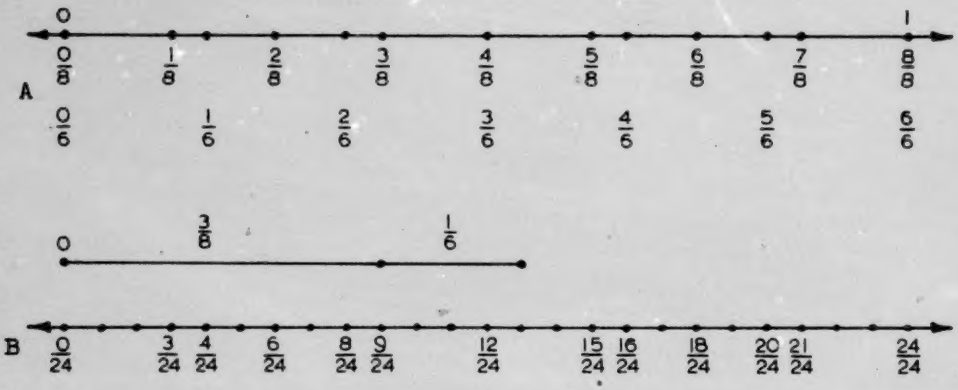
$$\text{l.c.d.} = \mathrlap{\overbrace{}^{8}}2 \times 2 \times \underbrace{2 \times 3}_{6} = 2 \times 2 \times 2 \times 3 = 24$$

To subdivide the eighths and sixths segments, note that

$$24 = 8 \times 3$$

$$24 = 6 \times 4$$

So each segment of measure one-eighth is subdivided into 3 congruent segments, and each segment of measure one-sixth is subdivided into 4 congruent segments.


0
1
A
0/8
1/8
2/8
3/8
4/8
5/8
6/8
7/8
8/8
0/6
1/6
2/6
3/6
4/6
5/6
6/6
0
3/8
1/6
B
0/24
3/24
4/24
6/24
8/24
9/24
12/24
15/24
16/24
18/24
20/24
21/24
24/24

On number line A the eighths and sixths scales are labeled. On B, points on the number line are marked for a scale of twenty-fourths. The points corresponding to eighths and sixths are labeled in twenty-fourths.

We write:

$$\frac{3}{8} + \frac{1}{6} = \frac{9}{24} + \frac{4}{24} = \frac{9 + 4}{24} = \frac{13}{24}.$$

Exercise Set 17

1. Find the scale with the smallest number of divisions you could use to picture these sums.

a. $\frac{3}{8} + \frac{7}{10}$ (fortieths)

b. $\frac{5}{9} + \frac{11}{15}$ (forty-fifths)

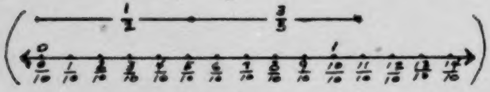
c. $\frac{3}{14} + \frac{5}{21}$ (forty-seconds)

d. $\frac{7}{16} + \frac{5}{12}$ (forty-eighths)

2. Use number line diagrams to picture:

a. $\frac{1}{2} + \frac{3}{5}$

b. $\frac{1}{6} + \frac{3}{4}$


1/2
3/5
0
1
0/10
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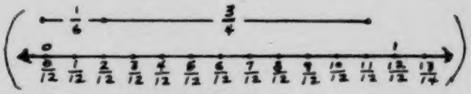

1/6
3/4
0
1
0/12
1/12
2/12
3/12
4/12
5/12
6/12
7/12
8/12
9/12
10/12
11/12
12/12
13/12

3. For each pair of fractions, write:

(1) the complete factorization of each denominator.

(2) the complete factorization of the least common denominator.

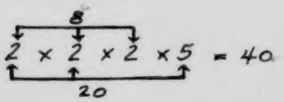
(3) fraction names using the l.c.d.

a. $\frac{3}{8}, \frac{7}{20}$

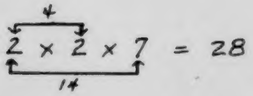
b. $\frac{1}{4}, \frac{3}{14}$

c. $\frac{2}{3}, \frac{4}{15}$

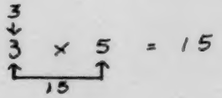
d. $\frac{5}{12}, \frac{7}{18}$

(a. 

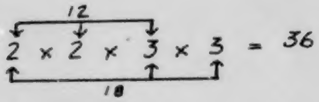
$2 \times 2 \times 2 \times 5 = 40$ $\frac{3}{8} = \frac{15}{40}$; $\frac{7}{20} = \frac{14}{40}$)

(b. 

$2 \times 2 \times 7 = 28$ $\frac{1}{4} = \frac{7}{28}$; $\frac{3}{14} = \frac{6}{28}$)

(c. 

$3 \times 5 = 15$ $\frac{2}{3} = \frac{10}{15}$; $\frac{4}{15} = \frac{4}{15}$)

(d. 

$2 \times 2 \times 3 \times 3 = 36$ $\frac{5}{12} = \frac{15}{36}$; $\frac{7}{18} = \frac{14}{36}$)

COMPUTING SUMS AND UNKNOWN ADDENDS

Objectives: To develop computational procedures for adding and subtracting rational numbers named by fractions with different denominators.

To show that such procedures reduce to (a) renaming the numbers by fractions with a common denominator and (b) adding or subtracting whole numbers.

Suggested Teaching Procedures

Work through the Exploration with the class, and work out with them the form in which written work is to be arranged.

Exercise Set 18 contains a list of exercises which pupils should now be able to do without using paper and pencil. Similar lists should be used frequently. Pupils really enjoy making up such lists for use by the class.

COMPUTING SUMS AND UNKNOWN ADDENDS

Exploration

You have seen that suitable scales can be found for picturing addition of rational numbers on the number line.

Rational numbers also can be added without using diagrams. Consider the sum

$$\frac{1}{2} + \frac{3}{4}.$$

You know how to add rational numbers when they are named by fractions with the same denominator.

1. What is the l.c.d. for $\frac{1}{2}$ and $\frac{3}{4}$?

2. Using the l.c.d., what are the fraction names for $\frac{1}{2}$ and $\frac{3}{4}$?

3. What is the sum of $\frac{1}{2}$ and $\frac{3}{4}$?

You can arrange your work like this:

$$n = \frac{1}{2} + \frac{3}{4}$$

$$= \frac{2}{4} + \frac{3}{4} = \frac{2+3}{4} = \frac{5}{4}$$

$$\frac{1}{2} + \frac{3}{4} = \frac{5}{4}$$

$$\text{l.c.d.} = \underbrace{\overset{\overset{2}{\downarrow}}{2} \times 2}_{4} = 4$$

$$\frac{1}{2} = \frac{2}{4}$$

4. Explain each step of the work (write complete factorization of each denominator. Find least common denominator. Rename fractions using least common denominator)

5. Can you find a common denominator for any two fractions? (yes)

6. Can you add any two rational numbers if you have fraction names for them? (yes)

Exercise Set 18

1. Find the sum of each pair of numbers. Write the simplest fraction name for the sum.

a. $\frac{1}{2}, \frac{3}{8}$ $\left(\frac{7}{8}\right)$ d. $\frac{2}{3}, \frac{1}{6}$ $\left(\frac{5}{6}\right)$ g. $\frac{3}{4}, \frac{5}{8}$ $\left(\frac{11}{8}\right)$

b. $\frac{1}{2}, \frac{5}{6}$ $\left(\frac{4}{3}\right)$ e. $\frac{2}{3}, \frac{5}{9}$ $\left(\frac{11}{9}\right)$ h. $\frac{3}{4}, \frac{17}{12}$ $\left(\frac{13}{6}\right)$

c. $\frac{1}{2}, \frac{7}{12}$ $\left(\frac{13}{12}\right)$ f. $\frac{2}{3}, \frac{11}{12}$ $\left(\frac{19}{12}\right)$ i. $\frac{3}{4}, \frac{27}{20}$ $\left(\frac{21}{10}\right)$

2. Find the simplest fraction name for each sum.

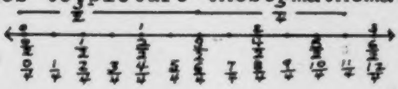
a. $\frac{3}{2} + \frac{1}{3}$ $\left(\frac{11}{6}\right)$ d. $\frac{8}{5} + \frac{7}{4}$ $\left(\frac{67}{20}\right)$ g. $\frac{5}{6} + \frac{3}{4}$ $\left(\frac{19}{12}\right)$

b. $\frac{3}{2} + \frac{4}{5}$ $\left(\frac{23}{10}\right)$ e. $\frac{8}{8} + \frac{2}{3}$ $\left(\frac{5}{3}\right)$ h. $\frac{5}{6} + \frac{3}{5}$ $\left(\frac{43}{30}\right)$

c. $\frac{3}{2} + \frac{6}{7}$ $\left(\frac{33}{14}\right)$ f. $\frac{8}{5} + \frac{2}{3}$ $\left(\frac{34}{15}\right)$ i. $\frac{5}{6} + \frac{1}{7}$ $\left(\frac{41}{42}\right)$

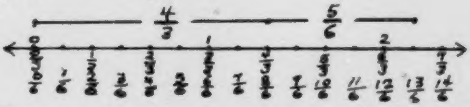
3. Use number lines to picture these mathematical sentences.

a. $\frac{3}{2} + \frac{5}{4} = n$

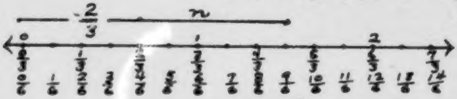


$\left(\frac{3}{2} + \frac{5}{4} = \frac{11}{4}\right)$

b. $\frac{4}{3} + \frac{5}{6} = n$ $\left(\frac{13}{6}\right)$



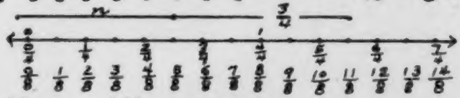
c. $\frac{2}{3} + n = \frac{3}{2}$



$\left(\begin{array}{l}\frac{2}{3} + n = \frac{3}{2} \\ \frac{4}{6} + \frac{5}{6} = \frac{3}{2} \\ n = \frac{5}{6}\end{array}\right)$

d. $\frac{11}{8} - \frac{3}{4} = n$

$\left(\begin{array}{l}n + \frac{3}{4} = \frac{11}{8} \\ n = \frac{5}{8}\end{array}\right)$



Rename each pair of numbers by fractions with a common denominator. Then find a fraction name for the number n.

4. a. $\frac{5}{6} - \frac{2}{3} = n$ $\left(\frac{1}{6}\right)$ d. $\frac{5}{6} - \frac{1}{3} = n$ $\left(\frac{1}{2}\right)$ g. $\frac{3}{4} - \frac{1}{6} = n$ $\left(\frac{7}{12}\right)$

b. $\frac{2}{3} - \frac{5}{8} = n$ $\left(\frac{1}{24}\right)$ e. $\frac{4}{5} - \frac{1}{3} = n$ $\left(\frac{7}{15}\right)$ h. $\frac{4}{5} - \frac{1}{2} = n$ $\left(\frac{3}{10}\right)$

c. $\begin{array}{r} \frac{11}{12} \\ -\frac{5}{6} \\ \hline n \end{array}$ $\left(\frac{1}{12}\right)$ f. $\begin{array}{r} \frac{7}{8} \\ -\frac{9}{16} \\ \hline n \end{array}$ $\left(\frac{5}{16}\right)$ i. $\begin{array}{r} \frac{3}{4} \\ -\frac{1}{8} \\ \hline n \end{array}$ $\left(\frac{5}{8}\right)$

Find the sum for each pair of numbers and write its simplest fraction name.

5. a. $\frac{1}{3}$, $\frac{3}{4}$ $\left(\frac{13}{12}\right)$ d. $\frac{3}{8}$, $\frac{1}{4}$ $\left(\frac{5}{8}\right)$ g. $\frac{2}{3}$, $\frac{5}{6}$ $\left(\frac{3}{2}\right)$

b. $\frac{3}{4}$, $\frac{1}{2}$ $\left(\frac{5}{4}\right)$ e. $\frac{2}{3}$, $\frac{1}{4}$ $\left(\frac{11}{12}\right)$ h. $\frac{4}{5}$, $\frac{1}{2}$ $\left(\frac{13}{10}\right)$

c. $\frac{2}{5}$ f. $\frac{3}{4}$ i. $\frac{1}{3}$

$\frac{1}{3}$ $\frac{1}{6}$ $\frac{1}{8}$

$\left(\frac{11}{15}\right)$ $\left(\frac{11}{12}\right)$ $\left(\frac{11}{24}\right)$

6. Find a fraction name for n so that each mathematical sentence will be true.

a. $\frac{2}{3} + n = \frac{3}{4}$ $\left(\frac{1}{12}\right)$ d. $\frac{3}{8} + n = \frac{2}{3}$ $\left(\frac{7}{24}\right)$

b. $\frac{3}{4} + n = \frac{7}{8}$ $\left(\frac{1}{8}\right)$ e. $\frac{1}{2} + n = \frac{2}{3}$ $\left(\frac{1}{6}\right)$

c. $\frac{2}{5} + n = \frac{3}{4}$ $\left(\frac{7}{20}\right)$ f. $\frac{1}{5} + n = \frac{1}{2}$ $\left(\frac{3}{10}\right)$

7. BRAINTWISTER: Find a fraction which names a number (Answers may vary.)

a. greater than $\frac{3}{10}$ and less than $\frac{3}{8}$. $\left(\frac{13}{40}, \frac{14}{40}\right)$

b. greater than $\frac{2}{6}$ and less than $\frac{3}{6}$. $\left(\frac{5}{12}, \frac{7}{18}, \frac{8}{18}\right)$

c. greater than $\frac{3}{6}$ and less than $\frac{4}{6}$. (Find two answers) $\left(\frac{7}{12}, \frac{10}{18}, \frac{11}{18}\right)$

Exercise Set 19

Certain rational numbers should now be so familiar that you can think of many names for them. You should be able to add and subtract such numbers without writing out your work.

Without doing any writing, try to find what number n must be.

1. $\frac{1}{4} + \frac{1}{8} = n$ $\left(\frac{3}{8}\right)$
2. $\frac{1}{2} + \frac{1}{5} = n$ $\left(\frac{7}{10}\right)$
3. $\frac{3}{4} + \frac{1}{2} = n$ $\left(\frac{5}{4}\right)$
4. $\frac{1}{6} + \frac{5}{12} = n$ $\left(\frac{7}{12}\right)$
5. $\frac{7}{8} + \frac{1}{4} = n$ $\left(\frac{9}{8}\right)$
6. $n + \frac{5}{8} = \frac{3}{4}$ $\left(\frac{1}{8}\right)$
7. $n + \frac{3}{10} = \frac{4}{5}$ $\left(\frac{1}{2}\right)$
8. $\frac{5}{6} + n = \frac{13}{12}$ $\left(\frac{1}{4}\right)$
9. $n = \frac{7}{9} - \frac{1}{3}$ $\left(\frac{4}{9}\right)$
10. $\frac{10}{10} = n - \frac{7}{10}$ $\left(\frac{17}{10}\right)$

The numbers in the exercises below have fraction names which are probably less familiar. Show all your work for these exercises.

11. $\frac{5}{6} + \frac{8}{15} = n$ $\left(\frac{41}{30}\right)$
12. $\frac{5}{14} + \frac{3}{4} = n$ $\left(\frac{31}{28}\right)$
13. $\frac{13}{12} + \frac{3}{8} = n$ $\left(\frac{35}{24}\right)$
14. $\frac{7}{10} + \frac{11}{6} = n$ $\left(\frac{38}{15}\right)$
15. $\frac{1}{6} + \frac{9}{14} = n$ $\left(\frac{17}{21}\right)$
16. $\frac{3}{10} + n = \frac{5}{4}$ $\left(\frac{19}{20}\right)$
17. $n + \frac{4}{9} = \frac{11}{12}$ $\left(\frac{17}{36}\right)$
18. $\frac{21}{10} = n - \frac{5}{8}$ $\left(\frac{109}{40}\right)$
19. $n + \frac{8}{15} = \frac{13}{9}$ $\left(\frac{41}{45}\right)$
20. $\frac{15}{7} = n + \frac{5}{4}$ $\left(\frac{25}{28}\right)$

Find the number n represents. In exercise 21, recall that $(\frac{1}{2} + \frac{1}{3}) + \frac{1}{4}$ means "Find the sum of $\frac{1}{2}$ and $\frac{1}{3}$, and then add the sum to $\frac{1}{4}$".

21. $(\frac{1}{2} + \frac{1}{3}) + \frac{1}{4} = n$ $\left(\frac{13}{12}\right)$
22. $(\frac{3}{4} + \frac{7}{8}) + \frac{2}{3} = n$ $\left(\frac{55}{24}\right)$
23. $\frac{5}{6} + (\frac{3}{2} + \frac{7}{4}) = n$ $\left(\frac{49}{12}\right)$
24. $n + (\frac{7}{12} + \frac{1}{4}) = \frac{5}{6}$ (0)
25. $(\frac{7}{10} + \frac{2}{5}) = n - \frac{1}{2}$ $\left(\frac{8}{5}\right)$
26. $(\frac{1}{4} + \frac{8}{9}) + n = \frac{5}{3}$ $\left(\frac{19}{36}\right)$

DIAGRAMS FOR PROBLEMS

Objectives: To suggest the use of the number line or a region to picture number relations in a problem.

To show that a number line diagram can be used to picture number relations in a problem even though the problem refers to objects which do not, of themselves, suggest segments.

Materials: Dittoed copies of number lines and regions.

Suggested Teaching Procedure:

Work through the exploration in the pupil text with the class. Have them make up additional problems solved by addition or subtraction with rational numbers and draw diagrams to picture the number relations involved.

Many children find it very difficult to solve "word" problems. For such children, thinking about the problem with sufficient care to draw a diagram which pictures the conditions can be very helpful. It is therefore sound procedure occasionally to require that they make diagrams for problems.

Some children enjoy drawing, and others find it a hard task. For this reason, it is a good idea to provide dittoed copies of number lines and circular and rectangular regions which they may use and adapt to their purposes. Models for some of these are included in the section on materials.

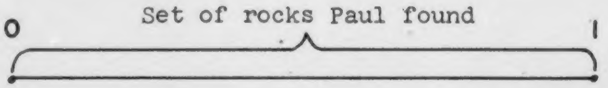
DIAGRAMS FOR PROBLEMS

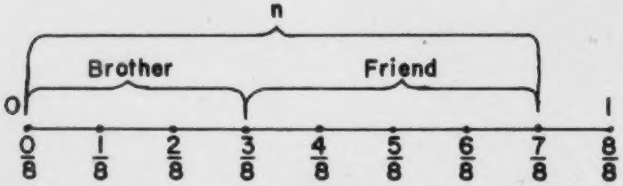
We have pictured addition and subtraction of rational numbers on the number line, and also with regions. When you wish to solve a problem using rational numbers, it is sometimes helpful to picture the relationships on a number line, or in a picture of a region. Look at this problem.

Paul found several unusual rocks while he was on vacation. He gave $\frac{3}{8}$ of the rocks to his brother, and gave $\frac{1}{2}$ of them to a friend. What part of the total number of rocks did he give away?

This is not a problem about things which suggest segments, but numbers are used, and numbers may be represented on the number line.

Suppose the unit segment represents the entire set of rocks Paul found.





He gave $\frac{3}{8}$ to his brother. He gave $\frac{1}{2}$ (or $\frac{4}{8}$) to a friend.

What part did he give away? Represent it by n. The diagram suggests:

$$n = \frac{3}{8} + \frac{1}{2}$$

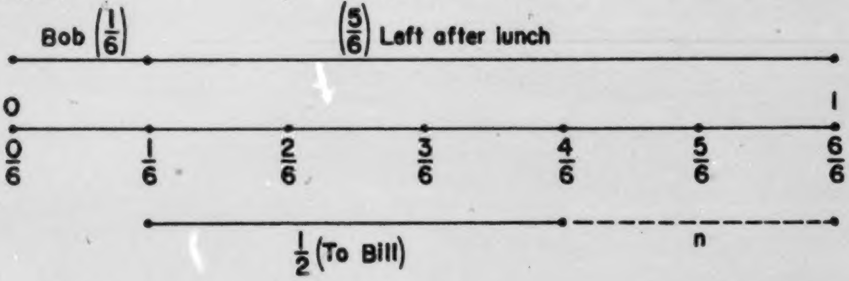
$$= \frac{3}{8} + \frac{4}{8} = \frac{3+4}{8} = \frac{7}{8}$$

Paul gave away $\frac{7}{8}$ of the rocks.

Now look at this problem.

Mrs. White cut a pie into 6 pieces. After Bob ate 1 piece for lunch, $\frac{5}{6}$ was left. Mrs. White served $\frac{1}{2}$ of the whole pie to Bill. What part of the pie was left then?

Choose a unit segment to represent the whole pie, cut into sixths.


Bob $\left(\frac{1}{6}\right)$
$\left(\frac{5}{6}\right)$ Left after lunch
0
1
$\frac{0}{6}$
$\frac{1}{6}$
$\frac{2}{6}$
$\frac{3}{6}$
$\frac{4}{6}$
$\frac{5}{6}$
$\frac{6}{6}$
$\frac{1}{2}$ (To Bill)
n

Bob ate $\frac{1}{6}$. Bill ate $\frac{1}{2}$ or $\frac{3}{6}$. What part of the pie was left?

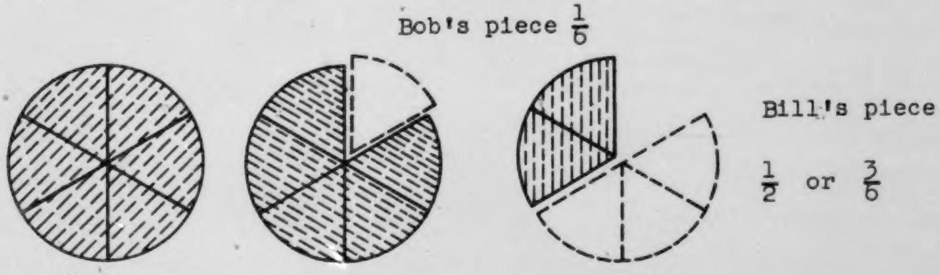
$$\frac{1}{2} + n = \frac{5}{6}, \text{ or}$$

$$\frac{3}{6} + n = \frac{5}{6}$$

$$n = \frac{5}{6} - \frac{3}{6} = \frac{5-3}{6} = \frac{2}{6}$$

$\frac{2}{6}$ of the pie was left.

You might wish to represent the whole pie as a unit region separated into sixths.


Bob's piece $\frac{1}{6}$
Bill's piece
$\frac{1}{2}$ or $\frac{3}{6}$

$1 = \frac{6}{6}$ $\qquad$ $1 - \frac{1}{6} = \frac{5}{6}$ $\qquad$ $\frac{5}{6} - \frac{3}{6} = \frac{2}{6}$

Exercise Set 20

For each problem, make a diagram showing the number relationships. Then write a mathematical sentence, and solve it. Write your answer to the problem in a sentence.

1. Susan bought $\frac{7}{8}$ lb. of fudge and $\frac{2}{8}$ lb. of chocolate drops. How much candy did she buy in all?
($\frac{7}{8} + \frac{2}{8} = n$; $n = \frac{9}{8}$. Susan bought $\frac{9}{8}$ lb in all.)

2. Tom and Jerry went to a Little League game. It took Tom $\frac{1}{4}$ hour to get to the game, and it took Jerry $\frac{3}{4}$ hour. How much longer did it take Jerry to get to the game than Tom? ($\frac{3}{4} - \frac{1}{4} = n$; $n = \frac{2}{4}$. It took Jerry $\frac{2}{4}$ hours longer.)

3. David caught a fish weighing $\frac{15}{16}$ lb. John's fish weighed $\frac{7}{16}$ lb. How much more did David's fish weigh?
$\frac{15}{16} - \frac{7}{16} = n$; $n = \frac{8}{16}$. David's fish weighed $\frac{8}{16}$ lb.)

4. Mrs. Ray had one whole coffee cake. She served $\frac{3}{8}$ of it to her neighbor. How much coffee cake did Mrs. Ray have left?
($1 - \frac{3}{8} = n$; $n = \frac{5}{8}$. She had $\frac{5}{8}$ cake left.)

5. Ann was mixing some punch for her friends. She mixed $\frac{2}{3}$ cup orange juice and $\frac{2}{3}$ cup gingerale. How much punch did she have?
($\frac{2}{3} + \frac{2}{3} = n$; $n = \frac{4}{3}$. She mixed $\frac{4}{3}$ cup.)

6. Mrs. King mixed some liquid plant food for her house plants. The directions said to use $\frac{3}{4}$ tablespoon for each gallon of water. She used 2 gallons of water. How much liquid plant food did Mrs. King use? ($\frac{3}{4} + \frac{3}{4} = n$; $n = \frac{6}{4}$. She used $\frac{6}{4}$ tablespoons plant food.)

7. Jack spent $\frac{3}{4}$ hour on Tuesday mowing the lawn. On Wednesday he spent $\frac{1}{2}$ hour pulling weeds. How much time did Jack spend doing his work? ($\frac{3}{4} + \frac{1}{2} = n$; $n = \frac{5}{4}$. He spent $\frac{5}{4}$ hours.)

8. Larry's mother gave him $\frac{1}{3}$ apple pie for his lunch. She gave his brother, Jim, $\frac{1}{6}$ of the same pie. How much of the pie did the two boys eat? ($\frac{1}{3} + \frac{1}{6} = n$; $n = \frac{1}{2}$. They ate $\frac{1}{2}$ pie.)

9. Janet bought $\frac{3}{4}$ yd. of material. She used $\frac{2}{3}$ yd. for place mats. How much material was left? ($\frac{3}{4} - \frac{2}{3} = n$; $n = \frac{1}{12}$. There was $\frac{1}{12}$ yard material left.)

10. Mrs. Smith used $\frac{7}{8}$ cup brown sugar and $\frac{1}{2}$ cup white sugar in a candy recipe. How much sugar did she use? ($\frac{7}{8} + \frac{1}{2} = n$; $n = \frac{11}{8}$. She used $\frac{11}{8}$ cup sugar.)

11. Alice stopped at the store on the way from her home to the park. It was $\frac{3}{5}$ mile to the store. The park was $\frac{9}{10}$ mile from Alice's home. How much farther did she walk to get to the park? ($\frac{9}{10} - \frac{3}{5} = n$; $n = \frac{3}{10}$. She walked $\frac{3}{10}$ mile farther.)

12. Jane spent $\frac{2}{3}$ hour doing her homework. Betty spent $\frac{1}{2}$ hour on homework. How much longer did it take Jane to finish? ($\frac{2}{3} - \frac{1}{2} = n$; $n = \frac{1}{6}$. It took her $\frac{1}{6}$ hr. longer.)

PROPERTIES OF ADDITION OF RATIONAL NUMBERS

Objective: To inquire whether the familiar properties of addition of whole numbers are true for addition of rational numbers.

Vocabulary: Associative Property for Addition, Commutative Property for Addition, Addition Property of Zero

Suggested Teaching Procedure

> To verify that the sum of two rational numbers is always a rational number, you may wish to proceed as follows:

We will try to find out if properties for the set of whole numbers under addition also hold for the set of rational numbers under addition.

Is the sum of two whole numbers always a whole number? (Yes) Give us some examples. (7 + 8 = 15, 11 + 14 = 25, 127 + 382 = 509)

Is the sum of two rational numbers always a rational number? (If the answer is "yes", ask for some examples.)

> Use a number of examples such as $\frac{1}{8} + \frac{1}{8}$, $\frac{5}{12} + \frac{2}{12}$, $\frac{3}{4} + \frac{1}{3}$, $\frac{11}{2} + \frac{8}{3}$, $\frac{5}{2} + \frac{2}{4}$. Lead pupils to see that in adding two rational numbers, such as $\frac{3}{4}$ and $\frac{1}{6}$, a common denominator for the fraction names can be found. Then the addition is reduced to one of adding whole numbers. The result of adding will still be a rational number.
>
> This discussion is concerned with what is technically called the closure property. We can say that the set of rational numbers is closed under the operation of addition. This language will probably not be used with children. They should understand it in the terms that the sum of any two rational numbers is a rational number.

Is it true that the sum of any two rational numbers is always one rational number? (Yes)

Does this property hold for subtraction? Let's look at some examples and see if we can answer that question.

Are the results of these subtractions rational numbers?

$$\frac{7}{4} - \frac{2}{4}, \quad \frac{11}{16} - \frac{5}{16}, \quad \frac{2}{3} - \frac{1}{2}, \quad \frac{2}{2} - \frac{3}{2}, \quad \frac{1}{4} - \frac{3}{4}$$

(The results of the first three are rational numbers. We do not yet have numbers for results of the last two. We know no rational number which added to $\frac{3}{2}$ has a result of $\frac{2}{2}$.)

Then will you agree that you cannot always subtract two rational numbers? (Yes)

> To verify that the Addition Property of Zero is true for rational numbers, you may wish to proceed in this way:

Think of two addends that are whole numbers. The sum of these two addends is one of the addends. What are some examples? ($6 + 0 = 6$, or $0 + 15 = 15$.) Is it true that if 0 is added to any whole number, the result is that whole number? (Yes) Does this property hold for rational numbers? (Yes) Give me some examples.

> Investigate with the class a number of examples such as $\frac{0}{2} + \frac{1}{2}$, $\frac{7}{3} + \frac{0}{3}$, and $\frac{9}{17} + \frac{0}{3}$.
>
> Here, it may be shown that $\frac{7}{3} + \frac{0}{3} = \frac{7}{3}$ by this reasoning.
>
> $$\begin{aligned}\frac{7}{3} + \frac{0}{3} &= \frac{7 + 0}{3}\\ &= \frac{7}{3}\end{aligned}$$

If $\frac{1}{2} + n = \frac{1}{2}$, what number is n? (0) If $n + \frac{13}{5} = \frac{13}{5}$, what number is n? (0) Can n be any other number except 0? (No)

> The Commutative Property for Addition may be discussed as follows:

Is it true that $7 + 9 = 9 + 7$? Is it true that $26 + 41 = 41 + 26$? (Yes) Try to state the rule for us. (Changing the order of adding two whole numbers does not change the sum.)

> This commutative property for addition of whole numbers is developed in grade 4. The word "commutative" is used with pupils to name this property.

This is called the commutative property for addition of whole numbers. Which of these mathematical sentences are true because of the commutative property for addition?

(a) $128 + 0 = 0 + 128$

(b) $256 + 891 = 891 + 256$

(c) $n + 621 = 621 + n$ (if $\underline{n}$ is any whole number)

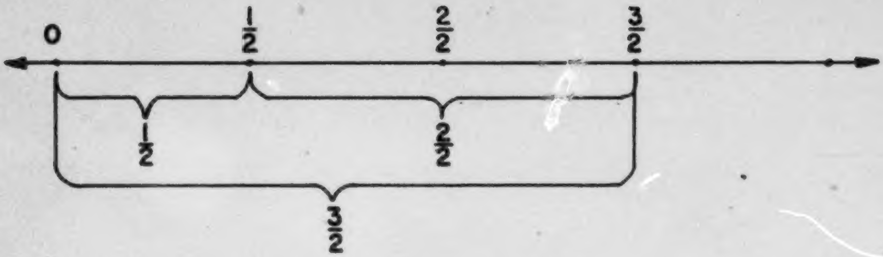
(d) $a + b = b + a$ (if $\underline{a}$ and $\underline{b}$ are any whole numbers)

(All are true by the commutative property for addition.)

Does this property hold for the addition of two rational numbers? (Yes) Are you sure?

> Have pupils show that sums $\frac{5}{8} + \frac{2}{8}$ and $\frac{2}{8} + \frac{5}{8}$ are the same. Continue with more examples until pupils understand, generalize, and freely use the commutative property.

Does the commutative property hold for the subtraction of two rational numbers? If you don't think so, how many examples do we need to find? (One) Give us an example. ($\frac{3}{2} - \frac{1}{2}$ is not equal to $\frac{1}{2} - \frac{3}{2}$.) Show this on the number line.



This shows $\frac{1}{2} + \frac{2}{2} = \frac{3}{2}$; therefore, $\frac{3}{2} - \frac{1}{2} = \frac{2}{2}$.

(We cannot show $\frac{1}{2} - \frac{3}{2}$ on the number line. To find $\frac{1}{2} - \frac{3}{2}$, think $\frac{1}{2} - \frac{3}{2} = n$. $\frac{1}{2}$ is the sum and $\frac{3}{2}$ one addend. The addend is greater than the sum. We cannot name a rational number n such that $n + \frac{3}{2} = \frac{1}{2}$.)

To verify the Associative Property:

Addition is an operation on two numbers to obtain a result of one and only one number. How do we add three rational numbers? To answer this question, we might think of how you added three whole numbers. How could you add 6, 7, and 8, in that order? (We could think $6 + 7 = 13$; $13 + 8 = 21$.) Tell us another way. (To 6, add the sum of 7 and 8. $6 + 15 = 21$.) Is the sum the same in both cases? (Yes)

What does $(6 + 7) + 8$ mean? (Add $6 + 7$ first. Add 8 to that sum.) What does $6 + (7 + 8)$ mean? (Add $7 + 8$ first. Add that sum to 6.)

Write a statement of this associative property for addition on the board: If three whole numbers are to be added, the third added to the sum of the first two is equal to the sum of the last two added to the first.

This is a long statement. Use a number of illustrations. Arrange your work on the board like this

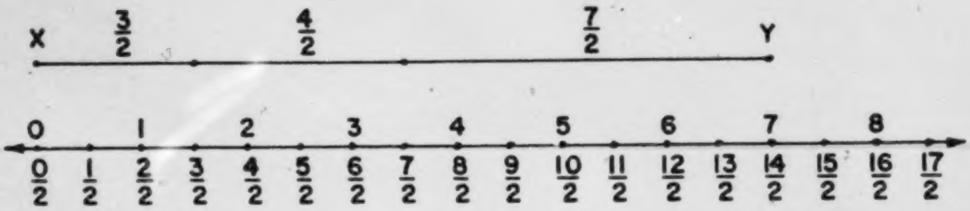
$$\text{Is} \quad (8 + 12) + 11 = 8 + (12 + 11)?$$

$$20 + 11 = 8 + 23$$

$$31 = 31$$

Do you think the associative property of addition holds for rational numbers? (Yes) Are you sure? Let us try some examples.

Show us $\frac{3}{2} + \frac{4}{2} + \frac{7}{2}$ on the number line.



Will the measure of $\overline{XY}$ be $\frac{14}{2}$ if we think $(\frac{3}{2} + \frac{4}{2}) + \frac{7}{2}$? (Yes)

Will the measure of $\overline{XY}$ be $\frac{14}{2}$ if we think $\frac{3}{2} + (\frac{4}{2} + \frac{7}{2})$? (Yes)

Use a number of examples of the associative property for addition with rational numbers as addends. This form is suitable for computation.

Is $(\frac{5}{2} + \frac{6}{2}) + \frac{3}{2} = \frac{5}{2} + (\frac{6}{2} + \frac{3}{2})$?

Does $\frac{5 + 6}{2} + \frac{3}{2} = \frac{5}{2} + \frac{6 + 3}{2}$?

Notice that

$$(5 + 6) + 3 = 5 + (6 + 3)$$

So $\frac{(5 + 6) + 3}{2} = \frac{5 + (6 + 3)}{2}$

$$\frac{11 + 3}{2} = \frac{5 + 9}{2}$$

$$\frac{14}{2} = \frac{14}{2}$$

The addition of rational numbers has the Associative Property. We see that this is true because of the fact that addition of whole numbers has this property.

PROPERTIES OF ADDITION OF RATIONAL NUMBERS

Exploration

You know that (1) any fraction, such as $\frac{2}{3}$, $\frac{8}{4}$, $\frac{50}{78}$, $\frac{5}{2}$, $\frac{0}{16}$, names a rational number. (2) You can find the point on a number line matching a rational number by (a) separating the unit segment into the number of congruent segments named by the denominator, (b) counting off from 0 the number of segments named by the numerator. (3) Some of these rational numbers, such as $\frac{6}{2}$, are also whole numbers.

1. Which numbers named above are also whole numbers? ($\frac{8}{4}$, $\frac{6}{2}$, 0)

2. What whole number can be a numerator of a fraction, but not a denominator? (0)

3. Think of two whole numbers. Find their sum. What kind of number is the sum? (whole number)

4. Think of two rational numbers. Find their sum. What kind of number is the sum? (rational number)

5. Think of two rational numbers. Subtract the smaller from the greater. What kind of number is your answer? (rational number)

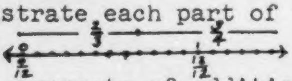
6. Try to subtract the greater number in exercise 5 from the smaller. Can you do it? (no) Can you always subtract one rational number from another? (no)

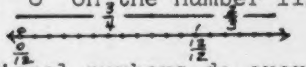
7. Think of a rational number n named by a fraction with denominator 6. Add it to $\frac{0}{6}$. What do you notice about the sum? (It is the same as the first addend.)

8. Find these sums.

a. $\frac{2}{3} + \frac{3}{4}$ $\left(\frac{17}{12}\right)$ b. $\frac{3}{4} + \frac{2}{3}$ $\left(\frac{17}{12}\right)$

9. Illustrate each part of exercise 8 on the number line.





10. What property of addition of rational numbers do exercises 8 and 9 suggest? (Commutative)

11. Find fraction names for the numbers n and t.

a. $(\frac{1}{2} + \frac{1}{3}) + \frac{3}{4} = n$ $\left(n = \frac{19}{12}\right)$ b. $\frac{1}{2} + (\frac{1}{3} + \frac{3}{4}) = t$ $\left(t = \frac{19}{12}\right)$

12. What do you notice about n and t in exercise 11? (They are the same)

13. What property of addition of rational numbers does exercise 11 suggest? (Associative)

Summary of Properties of Addition of Rational Numbers

1. If two rational numbers are added, the sum is a rational number.

2. If 0 is added to any rational number n, the sum is the same rational number n (Addition Property of Zero).

3. The order of adding two rational numbers may be changed without changing their sum (Commutative Property).

4. To find the sum of three rational numbers, you may (1) add the first two and add the third to their sum; or (2) add the second and third, and add their sum to the first (Associative Property).

Exercise Set 21

Write a mathematical sentence for each problem. Solve it and answer the question in a sentence.

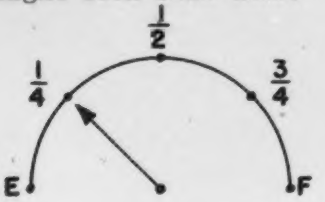
1. One measuring cup contains $\frac{1}{8}$ cup of liquid. A second measuring cup contains $\frac{3}{4}$ cup of liquid. If the liquid in the first cup is poured into the second cup, what amount of liquid will be in the second cup? ($\frac{1}{8} + \frac{3}{4} = n$; $n = \frac{7}{8}$. There will be $\frac{7}{8}$ cup liquid in second cup)

2. Directions on a can of concentrated orange juice call for mixing the juice with water. One-half quart water is to be mixed with $\frac{3}{16}$ quart of concentrated juice. What amount of liquid will result? ($\frac{1}{2} + \frac{3}{16} = n$; $n = \frac{11}{16}$. There will be $\frac{11}{16}$ qt. liquid)

3. I have $\frac{2}{3}$ dozen cookies in one box and $\frac{1}{4}$ dozen in another. You have 1 dozen cookies. Who has more? (you) How many cookies do I have? ($\frac{2}{3} + \frac{1}{4} = n$; $n = \frac{11}{12}$. I have 11 cookies)

4. A measuring cup is filled to the $\frac{3}{8}$ mark with milk. Enough water is added to bring the level of the mixture to the $\frac{3}{4}$ mark. How much water was added? ($\frac{3}{8} + n = \frac{3}{4}$; $n = \frac{3}{8}$. $\frac{3}{8}$ cup water was added)

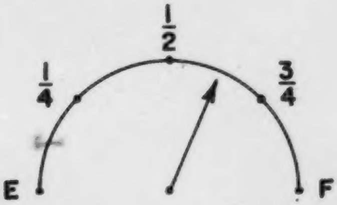
5. Cars have dials which show the quantity of gasoline in the tank. The dial might look like this:


$\frac{1}{2}$
$\frac{1}{4}$
$\frac{3}{4}$
E
F

How do the markings on the dial differ from those on our number lines? (They are not horizontal)

What unit of measure is represented on the dial? (one tank full)

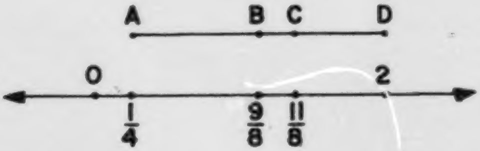
Suppose enough gasoline were added to the tank to move the pointer to a position halfway between the $\frac{1}{2}$ mark and the $\frac{3}{4}$ mark.


$\frac{1}{2}$
$\frac{1}{4}$
$\frac{3}{4}$
E
F

How much gasoline was added? ($\frac{3}{8}$ of a tank)

6. On the number line below the unit represented is the inch. What is the measure in inches of each of the six line segments pictured?

$m\overline{AB} = \frac{7}{8}$
$m\overline{BC} = \frac{1}{4}$
$m\overline{CD} = \frac{5}{8}$
$m\overline{AC} = \frac{9}{8}$
$m\overline{BD} = \frac{7}{8}$


A
B
C
D
O
2
$\frac{1}{4}$
$\frac{9}{8}$
$\frac{11}{8}$

WHOLE NUMBERS AND RATIONAL NUMBERS

FRACTIONS AND MIXED FORMS

RENAMING FRACTIONS IN MIXED FORM

Objectives: To develop the ideas that

(a) if the numerator of a fraction is greater than the denominator, the rational number it names may be renamed as a whole number or in mixed form.

(b) if the numerator of a fraction is a multiple of the denominator then the rational number it names is a whole number.

(c) a mixed form is a short form for an indicated sum, e.g. $2\frac{3}{4} = 2 + \frac{3}{4}$.

(d) every rational number either is a whole number or lies between two consecutive whole numbers.

To develop computational procedures for renaming rational numbers.

Vocabulary: Mixed form, simplest mixed form.

Materials: Number lines, cut-outs on flannel board, dittoed number lines for children.

Suggested Teaching Procedures

Use number lines and cut-outs on the flannel board to show that any rational number may be expressed as the sum of two rational numbers, one of which is a whole number (e.g., $\frac{2}{3} = 0 + \frac{2}{3}$, $\frac{5}{3} = 1 + \frac{2}{3}$, etc.) Some rational numbers may be expressed as whole numbers (e.g., $\frac{12}{3} = 4$) and this is the case when the numerator is a multiple of the denominator.

You may need to review division of whole numbers, and expression of a whole number n in the form $n = (d \times q) + r$, $r < d$.

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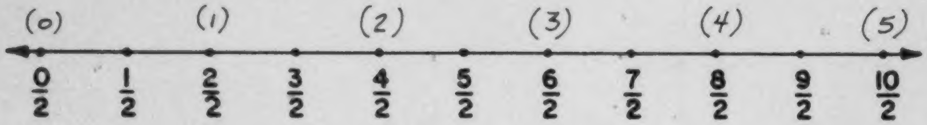
WHOLE NUMBERS AND RATIONAL NUMBERS

Exploration


(0)
(1)
(2)
(3)
(4)
(5)
$\frac{0}{2}$
$\frac{1}{2}$
$\frac{2}{2}$
$\frac{3}{2}$
$\frac{4}{2}$
$\frac{5}{2}$
$\frac{6}{2}$
$\frac{7}{2}$
$\frac{8}{2}$
$\frac{9}{2}$
$\frac{10}{2}$

1. Trace the number line and copy the scale of halves.

2. Write a whole number scale above the line. Be sure to keep the same unit segment. What number is at $\frac{0}{2}$? (0)

3. The whole number 1 should be written at the point labeled $(\frac{2}{2})$.

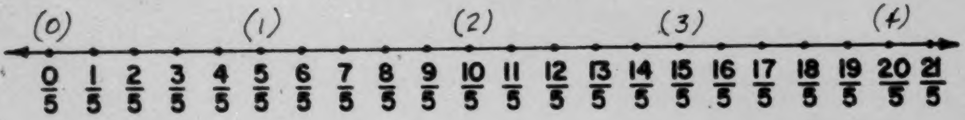
4. The whole number 2 should be written at the point labeled $(\frac{4}{2})$.

5. List the numerators of the fractions which name counting numbers. (2, 4, 6, 8, 10)

6. List the first five multiples of the denominator, 2.
(2, 4, 6, 8, 10)

7. Are your answers for exercises 5 and 6 the same? (yes)

8. Trace the number line and copy the scale of fifths.


(0)
(1)
(2)
(3)
(4)
0/5
1/5
2/5
3/5
4/5
5/5
6/5
7/5
8/5
9/5
10/5
11/5
12/5
13/5
14/5
15/5
16/5
17/5
18/5
19/5
20/5
21/5

9. Write a whole number scale above the line. Keep the same unit segment.

10. List the numerators of fractions which name counting numbers on the whole number scale. (5, 10, 15, 20)

11. List the first four multiples of the denominator, 5.
(5, 10, 15, 20)

12. Are your answers for exercises 10 and 11 the same? (yes)

13. Is this a true statement?

If the numerator of a fraction is a multiple of the denominator, then the fraction names a whole number. (yes)

14. Which of these fractions are names for whole numbers? (b,a,f)

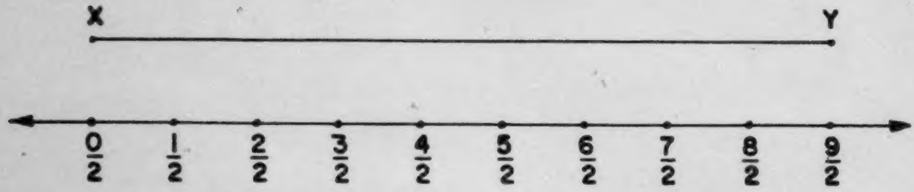
a. $\frac{18}{7}$ c. $\frac{60}{16}$ e. $\frac{120}{25}$

b. $\frac{27}{9}$ d. $\frac{90}{10}$ f. $\frac{28}{14}$

FRACTIONS AND MIXED FORMS

Exploration

Let us think of two rational numbers. Their sum is $\frac{9}{2}$. Here is a picture of $\overline{XY}$ measuring $\frac{9}{2}$. What could the addends be so their sum is $\frac{9}{2}$? (answers will vary)



1. Look at these pairs of addends. Is the sum of each pair $\frac{9}{2}$? (yes)

a. $\frac{8}{2} + \frac{1}{2}$	c. $\frac{6}{2} + \frac{3}{2}$
b. $\frac{2}{2} + \frac{7}{2}$	d. $\frac{4}{2} + \frac{5}{2}$

2. Write each of the sums in exercise 1 as the sum of a whole number and a rational number. (a. $4 + \frac{1}{2}$ c. $3 + \frac{3}{2}$ b. $1 + \frac{7}{2}$ d. $2 + \frac{5}{2}$)

3. Sums like your answers for exercise 2 are often written without the + sign, in this way:

a. $4\frac{1}{2}$ b. $1\frac{7}{2}$ c. $3\frac{3}{2}$ d. $2\frac{5}{2}$

Write these numbers as sums using the + sign.

e. $2\frac{1}{2}$	f. $3\frac{1}{2}$	g. $\frac{1}{2}$	h. $1\frac{1}{2}$
$(2 + \frac{1}{2})$	$(3 + \frac{1}{2})$	$(0 + \frac{1}{2})$	$(1 + \frac{1}{2})$

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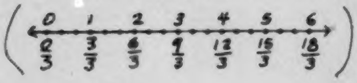
4. Think of the way you picture $2 + \frac{1}{2}$ with segments on the number line.

   a. What fractions of the halves scale name the endpoints of the segment with measure 2 ? $(\frac{0}{2}$ and $\frac{4}{2})$

   b. What fractions name the endpoints of the segment with measure $2\frac{1}{2}$? $(\frac{0}{2}$ and $\frac{5}{2})$

   c. What fraction names the same number as $2\frac{1}{2}$? $(\frac{5}{2})$

5. Use a number line. Write a whole number scale above it and a scale of thirds below it. Are these sentences true? (yes)


0 1 2 3 4 5 6
$\frac{0}{3}$ $\frac{3}{3}$ $\frac{6}{3}$ $\frac{9}{3}$ $\frac{12}{3}$ $\frac{15}{3}$ $\frac{18}{3}$

$$\begin{aligned} 5\frac{2}{3} &= 5 + \frac{2}{3} \\ &= \frac{5}{1} + \frac{2}{3} \\ &= \frac{5 \times 3}{1 \times 3} + \frac{2}{3} \\ &= \frac{15}{3} + \frac{2}{3} \\ &= \frac{17}{3} \end{aligned}$$

6. Find fraction names for these numbers.

   a. $6\frac{3}{4}$ $(\frac{27}{4})$ b. $4\frac{5}{7}$ $(\frac{33}{7})$ c. $12\frac{4}{3}$ $(\frac{40}{3})$ d. $12\frac{2}{6}$ $(\frac{74}{6})$

   Numerals like $6\frac{3}{4}$, $4\frac{5}{7}$, and $12\frac{4}{3}$ name rational numbers. These numerals are called mixed forms. In $6\frac{3}{4}$ and $4\frac{5}{7}$, the fractions $\frac{3}{4}$ and $\frac{5}{7}$ are in simplest form and name numbers less than 1. We say $6\frac{3}{4}$ and $4\frac{5}{7}$ are simplest mixed forms for rational numbers. $12\frac{4}{3}$ is not in simplest mixed form, because $\frac{4}{3} > 1$. $12\frac{2}{6}$ is not in simplest mixed form, because $\frac{2}{6}$ is not in simplest form.

Exercise Set 22

1. Find fraction names for these numbers.

a. $8\frac{2}{9}$ $\left(\frac{74}{9}\right)$

b. $11\frac{3}{5}$ $\left(\frac{58}{5}\right)$

c. $16\frac{1}{2}$ $\left(\frac{33}{2}\right)$

d. $3\frac{1}{7}$ $\left(\frac{22}{7}\right)$

e. $1\frac{7}{9}$ $\left(\frac{16}{9}\right)$

f. 15 $\left(\frac{15}{1}\right)$

2. Separate these fractions into 3 sets as follows:

Set M is the set of fractions which name whole numbers.

Set P is the set of fractions which can be expressed as mixed forms.

Set R is the set of fractions which name numbers less than 1.

a. $\frac{30}{5}$ (M)

b. $\frac{56}{4}$ (M)

c. $\frac{72}{8}$ (M)

d. $\frac{75}{8}$ (P)

e. $\frac{120}{10}$ (M)

f. $\frac{127}{10}$ (P)

g. $\frac{32}{5}$ (P)

h. $\frac{4}{9}$ (R)

i. $\frac{54}{9}$ (M)

3. Suppose you have a whole number scale and a scale labeled in fractions on the same number line. Between what two whole number points will a point lie which is labeled with these fractions?

a. $\frac{7}{3}$ (2, 3)

b. $\frac{9}{4}$ (2, 3)

c. $\frac{12}{5}$ (2, 3)

d. $\frac{19}{8}$ (2, 3)

e. $\frac{38}{7}$ (5, 6)

f. $\frac{83}{4}$ (20, 21)

4. Express these mixed forms as fractions. Then find the indicated sums and addends.

a. $\frac{3}{5} + 4\frac{1}{5} = n \quad \left(\frac{24}{5}\right)$

b. $8\frac{2}{3} - \frac{7}{4} = n \quad \left(\frac{83}{12}\right)$

c. $3\frac{1}{7} = 2\frac{1}{2} + n \quad \left(\frac{9}{14}\right)$

d. $6\frac{1}{5} + 1\frac{3}{10} = n \quad \left(\frac{15}{2}\right)$

5. Which number of each pair is greater? Answer in a sentence, using > or < .

a. $\frac{17}{8}, \quad 2\frac{3}{8} \quad \left(\frac{17}{18} < 2\frac{3}{8}\right)$

b. $7\frac{1}{5}, \quad \frac{31}{4} \quad \left(7\frac{1}{5} < \frac{31}{4}\right)$

c. $6\frac{3}{7}, \quad \frac{45}{8} \quad \left(6\frac{3}{7} > \frac{45}{8}\right)$

d. $9\frac{7}{10}, \quad \frac{48}{5} \quad \left(9\frac{7}{10} > \frac{48}{5}\right)$

e. $14\frac{1}{2}, \quad \frac{42}{3} \quad \left(14\frac{1}{2} > \frac{42}{3}\right)$

f. $\frac{39}{13}, \quad 2\frac{8}{13} \quad \left(\frac{39}{13} > 2\frac{8}{13}\right)$

RENAMING FRACTIONS IN MIXED FORM

Exploration

When a number is named by a mixed form, such as $2\frac{3}{7}$, you know how to rename it in fraction form.

$$\begin{aligned} 2\frac{3}{7} &= \frac{2}{1} + \frac{3}{7} \\ &= \frac{2 \times 7}{1 \times 7} + \frac{3}{7} \\ &= \frac{14}{7} + \frac{3}{7} \\ &= \frac{17}{7} \\ 2\frac{3}{7} &= \frac{17}{7} \end{aligned}$$

When a number is named by a fraction, you can easily rename it in mixed form when the numerator and denominator are small enough to use the number facts you know and to think about the points on the number line.

$$\begin{aligned} \frac{15}{4} &= \frac{12 + 3}{4} \\ &= \frac{12}{4} + \frac{3}{4} \\ &= 3 + \frac{3}{4} \\ &= 3\frac{3}{4} \\ \frac{15}{4} &= 3\frac{3}{4} \end{aligned}$$

Let us see how you can rename a number when the numerator and denominator are greater; for example, $\frac{437}{16}$.

Since a fraction names a whole number when the numerator is a multiple of the denominator, think about the set of multiples of 16.

Multiples of 16 = {16, 32, 48, 64 ...}

You would have to find a good many multiples of 16 to find a multiple close to 437. So try another way to find a multiple close to 437.

Suppose you write 437 in the form

$$437 = (16 \times n) + r, \text{ where } r < 16.$$

You know you can find n and r by using division. So 437 can be renamed as follows:

$437 = (16 \times 27) + 5$

```
     27        16
16 )437      × 27
    32        112
    117       32
    112       432
      5
```

Explain why these sentences are true:

a) $\frac{437}{16} = \frac{(16 \times 27) + 5}{16}$

b) $= \frac{432 + 5}{16}$

c) $= \frac{432}{16} + \frac{5}{16}$

d) $= 27 + \frac{5}{16}$

e) $= 27\frac{5}{16}$

f) $\frac{437}{16} = 27\frac{5}{16}$

In line b) is it necessary to write (16 × 27) as 432? In c), you could write

$$\frac{437}{16} = \frac{16 \times 27}{16 \times 1} + \frac{5}{16}$$

$$= \frac{27}{1} + \frac{5}{16} \qquad \text{(Why?)}$$

$$= 27\frac{5}{16}.$$

Find mixed form names for these numbers. Write your work in the way shown above.

1. $\frac{97}{13}$ $\left(7\frac{6}{13}\right)$

2. $\frac{147}{23}$ $\left(6\frac{9}{23}\right)$

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Exercise Set 23

Rename these numbers in simplest mixed form or as whole numbers. Show your work.

1. $$\frac{34}{5} = \frac{(5 \times 6) + 4}{5}$$

$$= \frac{5 \times 6}{5 \times 1} + \frac{4}{5}$$

$$= 6 + \frac{4}{5}$$

$$= 6\frac{4}{5}$$

$$\frac{34}{5} = 6\frac{4}{5}$$

2. $\frac{79}{8}$ $(9\frac{7}{8})$

3. $\frac{96}{11}$ $(8\frac{8}{11})$

4. $\frac{157}{13}$ $(12\frac{1}{13})$

5. $\frac{241}{15}$ $(16\frac{1}{15})$

6. $\frac{352}{7}$ $(50\frac{2}{7})$

7. $\frac{238}{10}$ $(23\frac{4}{5})$

8. $\frac{367}{12}$ $(30\frac{7}{12})$

9. $\frac{367}{36}$ $(10\frac{7}{36})$

10. $\frac{451}{100}$ $(4\frac{51}{100})$

11. $\frac{5280}{3}$ (1760)

Find simplest mixed forms to make these sentences true.

12. 50 ounces = $\frac{50}{16}$ pounds = $(3\frac{1}{8})$ pounds.

13. 100 feet = $\frac{100}{3}$ yards = $(33\frac{1}{3})$ yards.

14. 254 inches = $\frac{254}{12}$ feet = $(21\frac{1}{6})$ feet.

15. 37 pints = $\frac{37}{8}$ gallons = $(4\frac{5}{8})$ gallons.

16. Fill in the blanks in the table.

	Fraction	Simplest Fraction Name	Mixed Form
a.	$(\frac{62}{8})$	$(\frac{31}{4})$	$7\frac{6}{8}$
b.	$\frac{49}{10}$	$(\frac{49}{10})$	$(4\frac{9}{10})$
c.	$\frac{13}{11}$	$(\frac{13}{11})$	$(1\frac{2}{11})$
d.	$\frac{720}{25}$	$(\frac{144}{5})$	$(28\frac{4}{5})$
e.	$(\frac{323}{21})$	$(\frac{323}{21})$	$15\frac{8}{21}$
f.	$\frac{227}{10}$	$(\frac{227}{10})$	$22\frac{7}{10}$
g.	$\frac{79}{9}$	$(\frac{79}{9})$	$(8\frac{7}{9})$

COMPUTING WITH MIXED FORMS

Objective: To use the Commutative and Associative Properties to develop computational procedures for adding and subtracting rational numbers named by mixed forms.

Suggested Teaching Procedure

The development in the textbook provides an opportunity to emphasize the way in which the basic properties of rational numbers (Associative and Commutative) are applied in finding simple ways to compute sums.

You may wish also to have the pupils compute the sum of $4\frac{1}{2}$ and $7\frac{3}{4}$ by using the fraction names, $\frac{9}{2}$ and $\frac{31}{4}$, in order to verify that the result is the same as that shown.

Several forms which pupils may use for recording the steps in adding and subtracting rational numbers named by mixed forms are shown in the text. It is advisable for pupils to use the longer forms at first, and adopt shorter forms when the process is familiar and they can keep the steps in mind without recording all of them.

COMPUTING WITH MIXED FORMS

When numbers are expressed in mixed forms, you can add and subtract them without finding fraction names for them.

$$4\frac{1}{2} + 7\frac{3}{4} = n$$

a) $4\frac{1}{2} + 7\frac{3}{4} = (4 + \frac{1}{2}) + (7 + \frac{3}{4})$

b) $= 4 + (\frac{1}{2} + 7) + \frac{3}{4}$ (associative property)

c) $= 4 + (7 + \frac{1}{2}) + \frac{3}{4}$ (commutative property)

d) $= (4 + 7) + (\frac{1}{2} + \frac{3}{4})$ (associative)

e) $= 11 + (\frac{2}{4} + \frac{3}{4})$ Rename $\frac{1}{2}$; rename $(4+7)$

f) $= 11 + \frac{5}{4}$ (addition)

g) $= 11 + (\frac{4}{4} + \frac{1}{4})$ (renaming $\frac{5}{4}$)

h) $= 11 + (1 + \frac{1}{4})$ (renaming $\frac{4}{4}$)

i) $= (11 + 1) + \frac{1}{4}$ (associative)

j) $= 12 + \frac{1}{4}$ (renaming $11+1$)

k) $= 12\frac{1}{4}$ (renaming $12+\frac{1}{4}$)

Explain each line.

You do not need to write all this to show your work.

For example, write:

$$\begin{aligned} 4\frac{1}{2} + 7\frac{3}{4} &= (4 + 7) + (\frac{1}{2} + \frac{3}{4}) \\ &= 11 + (\frac{2}{4} + \frac{3}{4}) \\ &= 11 + \frac{5}{4} \\ &= 11 + 1 + \frac{1}{4} \\ &= 12\frac{1}{4} \end{aligned}$$

You may prefer this form:

$$\begin{array}{rl} 4\frac{1}{2} & = 4 + \frac{1}{2} = 4 + \frac{2}{4} \\ \underline{7\frac{3}{4}} & \underline{= 7 + \frac{3}{4}} \\ & 11 + \frac{5}{4} = 11 + 1 + \frac{1}{4} = 12\frac{1}{4} \end{array}$$

Or this:

$$\begin{array}{rcl} 4\frac{1}{2} & = & 4\frac{2}{4} \\ \underline{+\ 7\frac{3}{4}} & = & \underline{7\frac{3}{4}} \\ & & 11\frac{5}{4} = 12\frac{1}{4} \end{array}$$

The vertical form is best for subtraction. Explain these examples.

$$\begin{array}{rcl} 8\frac{5}{6} & = & 8\frac{10}{12} \\ \underline{-\ 3\frac{1}{4}} & = & \underline{3\frac{3}{12}} \\ & & 5\frac{7}{12} \end{array}$$

(common denominator)

$$\begin{array}{rcccccl} 7\frac{1}{6} & = & 7\frac{2}{12} & = & 6 + 1\frac{2}{12} & = & 6 + \frac{14}{12} \\ \underline{-\ 2\frac{3}{4}} & = & \underline{2\frac{9}{12}} & & & & \underline{2 + \frac{9}{12}} \\ & & & & & & 4 + \frac{5}{12} = 4\frac{5}{12} \end{array}$$

(common denominator plus renaming)

$$\begin{array}{rcccl} 15 & = & 14 + 1 & = & 14 + \frac{8}{8} \\ \underline{-\ 2\frac{5}{8}} & & & = & \underline{2 + \frac{5}{8}} \\ & & & & 12 + \frac{3}{8} = 12\frac{3}{8} \end{array}$$

(Renaming for subtraction)

Exercise Set 24

Find a fraction name or mixed form for n so each mathematical sentence is true. Show your work in the form your teacher suggests.

1. a. $4\frac{2}{3} + 2\frac{2}{3} = n$ $(7\frac{1}{3})$ b. $3\frac{2}{5} - 2\frac{1}{5} = n$ $(1\frac{1}{5})$ c. $1\frac{3}{8} + 2\frac{3}{8} = n$ $(3\frac{3}{4})$

2. a. $2\frac{1}{3} + 3\frac{1}{3} = n$ $(5\frac{2}{3})$ b. $3\frac{5}{6} - 1\frac{5}{6} = n$ (2) c. $3\frac{1}{5} + 2\frac{2}{5} = n$ $(5\frac{3}{5})$

3. a. $2\frac{1}{2} + 3\frac{3}{8} = n$ $(5\frac{7}{8})$ b. $3\frac{1}{2} + 2\frac{2}{5} = n$ $(5\frac{9}{10})$ c. $1\frac{3}{4} - 1\frac{1}{6} = n$ $(\frac{7}{12})$

4. a. $2\frac{3}{5} - 1\frac{1}{3} = n$ $(1\frac{4}{15})$ b. $4\frac{1}{2} - 2\frac{2}{3} = n$ $(1\frac{5}{6})$ c. $1\frac{7}{8} - 1\frac{1}{3} = n$ $(\frac{13}{24})$

5. a. $3\frac{1}{2} + 4\frac{1}{2} = n$ (8) b. $5\frac{2}{3} - 4\frac{1}{3} = n$ $(1\frac{1}{3})$ c. $1\frac{7}{12} - 1\frac{2}{12} = n$ $(\frac{5}{12})$

6. a. $4\frac{4}{5} - 3\frac{1}{2} = n$ $(1\frac{3}{10})$ b. $2\frac{1}{2} + 1\frac{7}{16} = n$ $(3\frac{15}{16})$ c. $3\frac{3}{4} + 2\frac{5}{8} = n$ $(6\frac{3}{8})$

7. a. $13 - 7\frac{5}{8} = n$ $(5\frac{3}{8})$ b. $8 - 2\frac{5}{16} = n$ $(5\frac{11}{16})$ c. $25 - \frac{4}{9} = n$ $(24\frac{5}{9})$

8. a. $12 - 7\frac{5}{8} = n$ $(4\frac{3}{8})$ b. $9 - 2\frac{5}{16} = n$ $(6\frac{11}{16})$ c. $18 - \frac{5}{4} = n$ $(16\frac{3}{4})$

9. a. $15\frac{7}{9} - 8 = n$ $(7\frac{7}{9})$ b. $15 - 8\frac{7}{9} = n$ $(6\frac{2}{9})$ c. $36 - \frac{11}{8} = n$ $(34\frac{5}{8})$

Exercise Set 25

Copy and subtract. Exercise 1 (a) is done for you.

1. a. $\begin{array}{r} 3\frac{6}{8} = 2 + \frac{14}{8} \\ \underline{1\frac{7}{8} = 1 + \frac{7}{8}} \\ 1 + \frac{7}{8} = 1\frac{7}{8} \end{array}$ b. $\begin{array}{r} 3\frac{3}{5} \\ \underline{1\frac{4}{5}} \\ (1\frac{4}{5}) \end{array}$ c. $\begin{array}{r} 4\frac{1}{6} \\ \underline{2\frac{5}{6}} \\ (1\frac{1}{3}) \end{array}$

2. a. $\begin{array}{r} 8\frac{3}{8} \\ \underline{4\frac{5}{8}} \\ (3\frac{3}{4}) \end{array}$ b. $\begin{array}{r} 7\frac{5}{12} \\ \underline{6\frac{7}{12}} \\ (\frac{5}{6}) \end{array}$ c. $\begin{array}{r} 8\frac{1}{3} \\ \underline{5\frac{2}{3}} \\ (2\frac{2}{3}) \end{array}$

3. a. $\begin{array}{r} 7\frac{1}{4} \\ \underline{3\frac{3}{4}} \\ (3\frac{1}{2}) \end{array}$ b. $\begin{array}{r} 9\frac{3}{10} \\ \underline{3\frac{7}{10}} \\ (5\frac{3}{5}) \end{array}$ c. $\begin{array}{r} 12\frac{2}{6} \\ \underline{3\frac{5}{6}} \\ (8\frac{1}{2}) \end{array}$

4. a. $\begin{array}{r} 5\frac{3}{8} \\ \underline{1\frac{7}{8}} \\ (3\frac{1}{2}) \end{array}$ b. $\begin{array}{r} 9\frac{1}{6} \\ \underline{3\frac{5}{6}} \\ (5\frac{1}{3}) \end{array}$ c. $\begin{array}{r} 4\frac{3}{16} \\ \underline{2\frac{9}{16}} \\ (1\frac{5}{8}) \end{array}$

5. a. $\begin{array}{r} 14\frac{1}{12} \\ \underline{9\frac{7}{12}} \\ (4\frac{1}{2}) \end{array}$ b. $\begin{array}{r} 9\frac{3}{10} \\ \underline{5\frac{9}{10}} \\ (3\frac{2}{5}) \end{array}$ c. $\begin{array}{r} 11\frac{7}{16} \\ \underline{8\frac{9}{16}} \\ (2\frac{7}{8}) \end{array}$

6. a. $\begin{array}{r} 5\frac{3}{8} \\ \underline{1\frac{5}{8}} \\ (3\frac{3}{4}) \end{array}$ b. $\begin{array}{r} 4\frac{1}{12} \\ \underline{2\frac{5}{12}} \\ (1\frac{2}{3}) \end{array}$ c. $\begin{array}{r} 4\frac{3}{10} \\ \underline{2\frac{9}{10}} \\ (1\frac{2}{5}) \end{array}$

7. BRAINTWISTER. Find n so each mathematical sentence is true.

a. $(11\frac{7}{8} - 4\frac{2}{8}) - 2\frac{3}{8} = n$ $(5\frac{1}{4})$ b. $(9\frac{7}{12} - 3\frac{1}{12}) + n = 10\frac{11}{12}$ $(4\frac{5}{12})$

ESTIMATING SUMS OF RATIONAL NUMBERS

Objectives: To reemphasize the fact that any rational number either (a) is a whole number, or (b) lies between two consecutive whole numbers.

To use this fact to determine two whole numbers which "bracket" the sum of two given rational numbers.

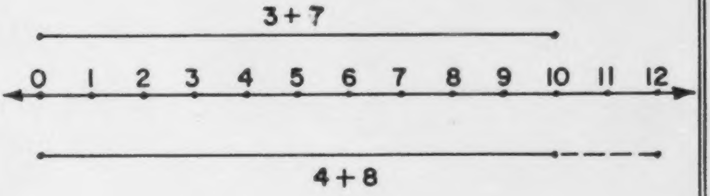
Vocabulary: >, <, consecutive.

Suggested Teaching Procedure

Before discussing "Estimating Sums of Rational Numbers", ask the pupils to tell whether suggested fractions and mixed forms name whole numbers; if not, to tell between what two consecutive whole numbers they lie. Use the number line if necessary.

A number line diagram can also be used to picture the interval within which the sum must be. For example, in $3\frac{4}{5} + 7\frac{2}{3}$, $3\frac{4}{5} > 3$ and $7\frac{2}{3} > 7$,

so $3\frac{4}{5} + 7\frac{2}{3} > 3 + 7$



$3\frac{4}{5} < 4$ and $7\frac{2}{3} < 8$, so $3\frac{4}{5} + 7\frac{2}{3} < 4 + 8$.

The dotted segment represents the interval within which the segment whose measure is $3\frac{4}{5} + 7\frac{2}{3}$ must end.

Pupils should be encouraged to use the procedure outlined in this section to judge the probable correctness of any sums or addends they compute.

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ESTIMATING SUMS OF RATIONAL NUMBERS

Exploration

When you are adding rational numbers, it is a good idea to estimate the sum first.

1) Consider the sentence:

$$3\frac{4}{5} + 7\frac{2}{3} = n$$

Between what two consecutive whole numbers would each addend be?

a. $3\frac{4}{5} >$ (3) and $3\frac{4}{5} <$ (4)

b. $7\frac{2}{3} >$ (7) and $7\frac{2}{3} <$ (8)

Are these statements true?

c. $3\frac{4}{5} + 7\frac{2}{3} > 3 + 7$ (yes)

d. $3\frac{4}{5} + 7\frac{2}{3} < 4 + 8$ (yes)

e. The sum of $3\frac{4}{5}$ and $7\frac{2}{3}$ is a number between 10 and 12. (yes)

2) Between what two consecutive whole numbers is

a. $\frac{17}{2}$? (8 and 9)

b. $\frac{13}{4}$? (3 and 4)

c. The sum of $\frac{17}{2}$ and $\frac{11}{4}$ must be a number greater than (11) and less than (13).

Exercise Set 26

Which of the answers below *may* be right? Which ones *must* be wrong? Answer by finding two consecutive whole numbers between which the sum must be.

1. $\frac{7}{8} + 1\frac{3}{4} = \frac{29}{8}$ (must be wrong)
2. $\frac{11}{3} + \frac{5}{6} = 4\frac{1}{2}$ (may be right)
3. $\frac{13}{2} + 7\frac{3}{4} = 15\frac{1}{4}$ (must be wrong)
4. $14\frac{4}{5} + 6\frac{9}{10} = 20\frac{7}{10}$ (may be right)
5. $\frac{25}{7} + \frac{14}{3} = 6\frac{5}{21}$ (wrong)
6. $\frac{21}{8} + \frac{19}{4} = \frac{59}{8}$ (right)

Between what two consecutive whole numbers must each sum be?

7. $2\frac{3}{4} + 3\frac{1}{2}$ (5, 7)
8. $7\frac{8}{9} + 14\frac{2}{3}$ (21, 23)
9. $10\frac{7}{10} + 12\frac{4}{5}$ (22, 24)
10. $128\frac{5}{6} + 73\frac{1}{8}$ (201, 203)
11. $64\frac{3}{8} + 19\frac{3}{4}$ (83, 85)
12. $89\frac{1}{7} + 15\frac{9}{10}$ (104, 106)

Might these addends be correct?

13. $5\frac{3}{7} - 2\frac{1}{2} = 2\frac{5}{14}$ (Think: If $2\frac{5}{14} + 2\frac{1}{2} = n$, then $n >$ (4) and $n <$ (6))
14. $12\frac{7}{12} - 5\frac{7}{8} = 4\frac{9}{16}$ (No)
15. $15 - 6\frac{3}{8} = 9\frac{3}{8}$ (No)

Exercise Set 27

1. Robert needs $\frac{22}{3}$ feet of new cord to reach from his desk lamp to a wall outlet. The hardware store sells lamp cord in no smaller divisions than the foot. How long a piece of lamp cord will Robert have to buy? ($\frac{22}{3} = 7\frac{1}{3}$. He will have to buy 8 feet.)

2. Joan's family leaves on a trip at noon. The time required for the round trip is $\frac{13}{2}$ hours. At what time will they be back? ($\frac{13}{2} = 6\frac{1}{2}$. They will be back at 6:30 pm.)

3. Suppose a man finds that to paint the outside of his house he will need about 17 quarts of paint. The paint he needs is sold only in gallons. How much paint will he need to buy? ($\frac{17}{4} = 4\frac{1}{4}$. He will have to buy 5 gallons.)

4. Driving time from Boston to New York is $\frac{9}{2}$ hours. Driving from New York to Philadelphia requires $\frac{7}{4}$ hours. How long does it take to drive from Boston to Philadelphia by way of New York? ($\frac{9}{2} = 4\frac{1}{2}$; $\frac{7}{4} = 1\frac{3}{4}$; $4\frac{1}{2} + 1\frac{3}{4} = n$; $n = 6\frac{1}{4}$. It will take $6\frac{1}{4}$ hr.)

A magic square is one in which you can perform the operation on the numbers vertically, horizontally or diagonally and always get the same number for a result.

5. Copy the square below.

   a. Add the numbers named by the fractions in each column and record the sum for each column. (10)

   b. Add the numbers named by the fractions in each row and record the sum for each row. (10)

   c. Begin in lower left hand corner. Add the numbers named by the fractions diagonally. Record their sum. (10)

   d. Begin in upper left hand corner. Add the numbers named by the fractions diagonally. Record their sum. (10)

   e. Is each sum the same rational number? What is the number? (10)

   f. Is the square a magic square? (yes)

$2\frac{1}{2}$	$3\frac{3}{8}$	$\frac{1}{2}$	$1\frac{3}{8}$	$2\frac{1}{4}$
$3\frac{1}{4}$	1	$1\frac{1}{4}$	$2\frac{1}{8}$	$2\frac{3}{8}$
$\frac{7}{8}$	$1\frac{1}{8}$	2	$2\frac{7}{8}$	$3\frac{1}{8}$
$1\frac{5}{8}$	$1\frac{7}{8}$	$2\frac{3}{4}$	3	$\frac{3}{4}$
$1\frac{3}{4}$	$2\frac{5}{8}$	$3\frac{1}{2}$	$\frac{5}{8}$	$1\frac{1}{2}$

6. Copy the square below. Write fractions in A, B, C, D, E, and F to make it a magic square whose sum is 5. (Recall that a "magic square" is one in which the sum of the numbers named in a row, a column, or on a diagonal is the same number. This number is called the "sum" for the square.)

A	B	$1\frac{11}{12}$
C	D	E
$1\frac{5}{12}$	F	$\frac{11}{12}$

A	$2\frac{5}{12}$
B	$\frac{2}{3}$
C	$1\frac{1}{6}$
D	$1\frac{2}{3}$
E	$2\frac{1}{6}$
F	$2\frac{2}{3}$

THINKING ABOUT DECIMALS

Objective: To develop understanding of the naming of rational numbers by using decimals, and ability to translate from fraction symbols to decimal symbols.

Materials: Place value pocket chart and cards.
Enlarged number line as shown in the Exploration

Suggested Teaching Procedure

Review the system of decimal notation. Use the place value pocket chart as needed. (See Grade 4, Chapter 2; Grade 5, Chapter 1.)

The pupils have studied some properties of rational numbers and have learned to add and subtract them, using fraction names. They should learn to look on decimals simply as other names for these familiar numbers.

Decimal language has certain advantages for computing which fraction language does not have In order to exploit these advantages, it is important that the pupils learn to think in more than one way about the numbers named by decimals. Start with decimal names for whole numbers.

345 is 3 hundreds + 4 tens + 5 ones
or 34 tens + 5 ones
or 3 hundreds + 45 ones, etc.

Similarly,

3.45 is 3 ones + 4 tenths + 5 hundredths
or 34 tenths + 5 hundredths
or 2 ones + 14 tenths + 5 hundredths, etc.

A number line showing tenths and hundredths, as shown in the Exploration, may be drawn on the board and used to advantage.

To simplify notation, the 0 before the decimal point in the decimal numeral for a number less than 1 is sometimes omitted.

THINKING ABOUT DECIMALS

Exploration

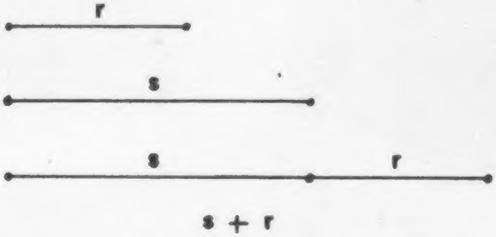
If you had grown in France, you would say

"Mon frere est plus grand que moi,"

instead of "My brother is taller than I."

Both sentences express the same relation. A French pupil does not have to know the English language to understand the relation we call "taller". We do not have to speak French to understand this relation. But to understand the idea in the French sentence we would have to translate it into English. As soon as we learned to understand French well, we would not have to translate French sentences. We would think in French.

Our problem with rational numbers is very much like this. We know a meaning for addition. If s and r are rational numbers, then $s + r$ can be pictured as measures in this way:



We also know how to express addition relations in the "language" of fractions, and in mixed form "language".

$$\frac{8}{5} + \frac{37}{10} = \frac{16}{10} + \frac{37}{10} = \frac{53}{10}$$

$$1\frac{3}{5} + 3\frac{7}{10} = 1 + 3 + \frac{6}{10} + \frac{7}{10} = 4 + \frac{13}{10} = 5\frac{3}{10}.$$

But these are not the only languages we use for rational numbers. A very common language is the decimal numeral system. We already know how to "translate" some fractions into decimals. For example:

$$\frac{6}{10} = .6, \quad \frac{5}{10} = .5.$$

In fraction language we can express this addition relation:

$$\frac{6}{10} + \frac{5}{10} = \frac{11}{10}.$$

Can we translate this sentence into decimal language?

We only have to translate $\frac{11}{10}$. To do this we first find a mixed form expression.

$$\frac{11}{10} = \frac{10 + 1}{10} = \frac{10}{10} + \frac{1}{10} = 1 + \frac{1}{10} = 1\frac{1}{10} = 1.1$$

Now we can write the decimal sentence:

$$.6 + .5 = 1.1$$

We can always use this method, but it can be long. Here is another example:

What is the decimal name for $.38 + .75$?

First we translate: $.38 = \frac{38}{100}$, $.75 = \frac{75}{100}$.

We compute: $\frac{38}{100} + \frac{75}{100} = \frac{113}{100}$.

We translate to mixed form: $\frac{113}{100} = 1\frac{13}{100}$.

We translate back: $1\frac{13}{100} = 1.13$

In decimal language:

$$.38 + .75 = 1.13.$$

Our problem is this. Can we learn to express addition relations in decimal language without translating to fractions and back? Yes, we can. It is really very easy. We can begin to learn how by thinking of the meaning of decimals.

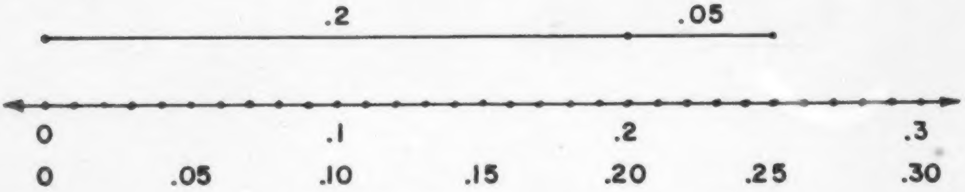
It helped us in thinking about decimal numerals for whole numbers to write an expanded form. Now we will write it this way:

$$\begin{aligned} 246 &= 200 + 40 + 6 \\ &= 2 \text{ hundreds} + 4 \text{ tens} + 6 \text{ ones}. \end{aligned}$$

Can we think of all decimals in the same way? Can we think of .25 and 8.4 and 1.06 in this way? We know

$$\begin{aligned} .25 = \frac{25}{100} &= \frac{20}{100} + \frac{5}{100} \\ &= \frac{2}{10} + \frac{5}{100} \\ &= .2 + .05 \\ &= 2 \text{ tenths} + 5 \text{ hundredths} \end{aligned}$$

A number line diagram helps us to understand why this is so. Look at this number line, marked with a tenths scale and a hundredths scale.



Now consider 8.4. We know $8.4 = 8\frac{4}{10} = 8 + \frac{4}{10} = 8 + .4$.

We know $1.06 = 1\frac{6}{100} = 1 + \frac{6}{100} = 1 + .06$.

Write these in expanded form:

1. 38.47 (30 + 8 + .4 + .07)

2. 160.13 (100 + 60 + .1 + .03)

3. 57.06 (50 + 7 + .06)

We can express the meaning of decimal numerals in many ways.

27.38 = 2 tens + 7 ones + 3 tenths + 8 hundredths
= 27 ones + 38 hundredths
= 270 tenths + 38 hundredths
= 2 tens + 7 ones + 2 tenths + 18 hundredths.
= 20 + 7 + .3 + .08

2738 = 2 thousands + 7 hundreds + 3 tens + 8 ones
= 27 hundreds + 38 ones
= 272 tens + 18 ones
= 2000 + 700 + 30 + 8

2.738 = 2 ones + 7 tenths + 3 hundredths + 8 thousandths
= 27 tenths + 38 thousandths
= 2 ones + 6 tenths + 13 hundredths + 8 thousandths
= 2 + .7 + .03 + .008

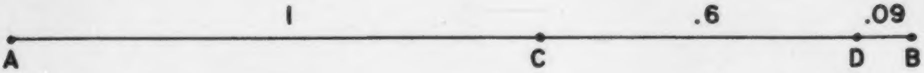
Rename these numbers four ways, as shown on the preceding page.

4. 6.84 (6 ones + 8 tenths + 4 hundredths
68 tenths + 4 hundredths
6 ones + 84 hundredths
6 + .8 + .04, etc.)

5. 68.4 (68 ones and 4 tenths
6 tens + 8 ones + 4 tenths
684 tenths
60 + 8 + .4, etc.)

6. 70.605 (7 tens + 0 ones + 6 tenths + 0 hundredths + 5 thousandths
70 ones + 6 tenths + 5 thousandths
7 tens + 60 hundredths + 5 thousandths
70 + .6 + .00 + .005, etc.)

7. What is the measure $\overline{AB}$ pictured below? (1.69) $\overline{AC}$ has measure 1, $\overline{CD}$ has measure .6 and $\overline{DB}$ has measure .09.



What is the decimal for each of these?

8. 5 tens + 6 ones + 4 tenths + 3 hundredths? (56.43)

9. 5 tenths + 6 tens + 4 ones + 13 hundredths? (64.63)

10. 6 tens + 6 hundredths. (60.06)

Translate to decimals:

11. $\frac{567}{1000}$ (.567)

12. $\frac{567}{100}$ (5.67)

13. $\frac{567}{10}$ (56.7)

14. $28\frac{34}{100}$ (28.34)

15. $40\frac{6}{1000}$ (40.006)

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Exercise Set 28

1. Complete:

a. 72.9 = (2) ones and (9) tenths and (7) tens.

b. 702.09 = (7) hundreds + (2) ones + (0) tenths + (0) tens + (9) hundredths.

c. 68.75 = (68) ones + (75) hundredths.

d. $52\frac{16}{100}$ = (52) ones + (16) hundredths.

e. $400 + 5 + \frac{15}{100}$ = (4) hundreds + (0) tens + (5) ones + (1) tenths + (5) hundredths.

2. Write the decimal for each of these.

a. 5 hundreds and 5 hundredths (500.05)

b. 43 tens + 16 hundredths (430.16)

c. 2 tens + 7 hundreds + 6 ones (726)

d. 14 hundredths + 6 tenths + 3 ones (3.74)

e. 12 hundredths + 9 tenths. (1.02)

3. Write the decimal for each of these.

a. $\frac{14}{10}$ (1.4)

b. $6\frac{3}{10}$ (6.3)

c. $\frac{5243}{1000}$ (5.243)

d. $\frac{5243}{10}$ (524.3)

e. $\frac{6840}{10}$ (684.0)

f. $20\frac{6}{100}$ (20.06)

4. Express in dollars the value of:

a. 4 ten dollar bills, 8 dimes, and 6 one dollar bills. ($46.80)

b. 15 one dollar bills, 12 pennies, and 7 dimes. ($15.82)

c. 253 pennies. ($2.53)

d. 8 one dollar bills and 58 pennies ($8.58)

BRAINTWISTERS:

5. Translate to decimals:

a. $\frac{4}{25}$ (.16)

b. $6\frac{7}{20}$ (6.35)

c. 3 fives + 20 fifths + 2 twenty-fifths (19.08)

d. $\frac{7}{8}$ (.875)

6. Which is greater?

a. .33 or $\frac{1}{3}$? ($\frac{1}{3}$. $33 = \frac{33}{100} = \frac{99}{300}$; $\frac{1}{3} = \frac{100}{300}$)

b. .125 or $\frac{1}{8}$? (neither)

c. .166 or $\frac{1}{6}$? ($\frac{1}{6}$; $.166 = \frac{166}{1000} = \frac{996}{6000}$; $\frac{1}{6} = \frac{1000}{6000}$)

ADDITION OF RATIONAL NUMBERS USING DECIMALS

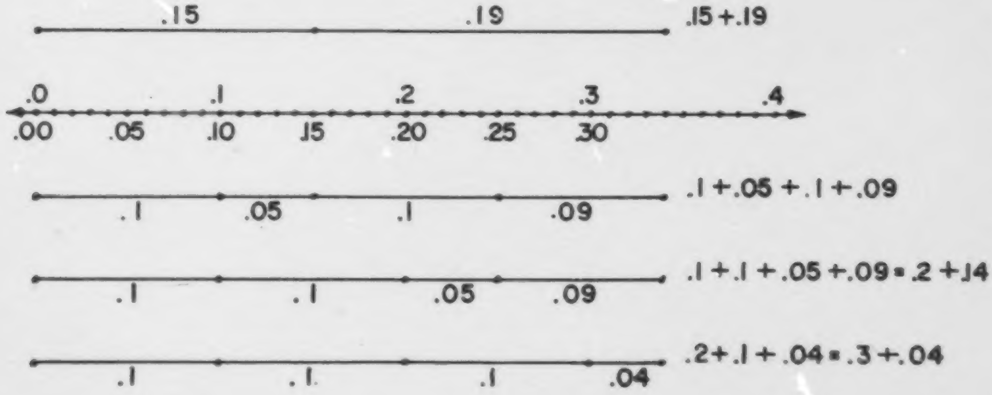
Objective: To use understanding of place value in the decimal system to develop a method for computing sums of rational numbers named by decimals.

Materials: Number lines sealed in tenths and in hundredths.

Suggested Teaching Procedure:

Since decimal notation for rational numbers has the same properties as decimal notations for whole numbers, the discussion of addition and subtraction of rational numbers using decimal numerals emphasizes these properties. The place value property of decimal numerals is used to name each number in expanded notation. The commutative and associative properties of addition are used to group tenths with tenths, hundredths with hundredths, etc. Place value is used again to regroup hundredths when a sum is more than 9 hundredths, tenths when a sum is more than 9 tenths, etc. Thus, all the properties used in computing sums of rational numbers using decimal numerals are already familiar.

The use of these properties in addition can be exhibited on a number line with scales in tenths and hundredths.



ADDITION OF RATIONAL NUMBERS USING DECIMALS

Exploration

Now we are ready to find a quick way to add or subtract rational numbers using decimal names.

Suppose we want to add .12 and .34. We could translate to fractions:

$$
\begin{aligned}
.12 + .34 &= \frac{12}{100} + \frac{34}{100} \\
&= \frac{12 + 34}{100} \\
&= \frac{46}{100} \\
&= .46
\end{aligned}
$$

But we could also remember this:

$$
\begin{aligned}
.12 + .34 &= (.1 + .02) + (.3 + .04) \\
&= (.1 + .3) + (.02 + .04)
\end{aligned}
$$

What properties of addition have we used? (Associative and Commutative)

Now $.1 + .3$ = 1 tenth + 3 tenths = 4 tenths

$.02 + .04$ = 2 hundredths + 4 hundredths = 6 hundredths

In decimals:

$$
\begin{aligned}
.1 + .3 &= .4 \\
.02 + .04 &= .06
\end{aligned}
$$

We have, then,

$$
\begin{aligned}
.12 + .34 &= (.1 + .02) + (.3 + .04) \\
&= (.1 + .3) + (.02 + .04) \\
&= .4 + .06 \\
&= .46
\end{aligned}
$$

This is not a new method. The place value idea is used for tenths and hundredths. We have already used this idea for places to the left of the decimal point.

Here is an example with numerals on both sides of the decimal point.

$$\begin{aligned} 16.31 + 43.52 &= (10 + 6 + .3 + .01) + (40 + 3 + .5 + .02) \\ &= (10 + 40) + (6 + 3) + (.3 + .5) + (.01 + .02) \\ &= 50 + 9 + .8 + .03 \\ &= 59.83 \end{aligned}$$

We can use the vertical form to make the computations easier:

$$\begin{array}{rl} 16.31 = & 10 + 6 + .3 + .01 \\ + \; 43.52 = & \underline{40 + 3 + .5 + .02} \\ & 50 + 9 + .8 + .03 = 59.83 \end{array}$$

Use the vertical form to compute:

(1) $18.5 + 31.4$ (49.9) (2) $72.06 + 15.43$ (87.49)

How should we think of problems like these?

(a) $.7 + .8$ (b) $.06 + .09$

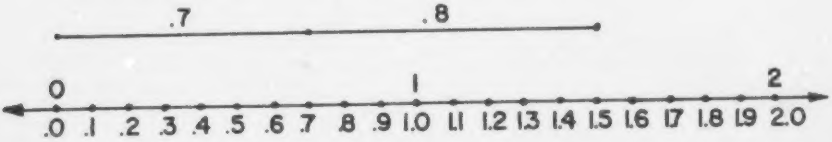
(a) We know: $.7 + .8 = 7$ tenths + 8 tenths = 15 tenths

= 10 tenths + 5 tenths

= 1 one + 5 tenths

= 1.5

You could picture $.7 + .8$ on the number line.



(b) We know:

$$\begin{aligned} .06 + .09 &= 6 \text{ hundredths} + 9 \text{ hundredths} \\ &= 15 \text{ hundredths} \\ &= 10 \text{ hundredths} + 5 \text{ hundredths} \\ &= 1 \text{ tenth} + 5 \text{ hundredths} \\ &= .15 \end{aligned}$$

This is the same idea, regrouping in sets of ten, that we have used many times in problems like:

$80 + 70 = 150$ or $600 + 900 = 1500$.

There is one thing new. We must be very careful to locate the decimal point correctly.

Now we can do problems like these:

$$\begin{aligned} 14.56 &= 10 + 4 + .5 + .06 \\ + 27.25 &= \underline{20 + 7 + .2 + .05} \\ & 30 + 11 + .7 + .11 = 30 + 11 + (.7 + .1) + .01 \\ & = (30 + 10) + 1 + .8 + .01 \\ & = 40 + 1 + .8 + .01 \\ & = 41.81 \end{aligned}$$

Try these examples:

(3) $\begin{array}{r} 6.37 \\ \underline{+ 3.24} \end{array}$

$$\left(\begin{array}{l} 6 + .3 + .07 \\ \underline{3 + .2 + .04} \\ 9 + .5 + .11 = \\ 9 + .5 + .1 + .01 = \\ 9 + .6 + .01 = 9.61 \end{array}\right)$$

(4) $\begin{array}{r} 20.08 \\ \underline{+ 7.39} \end{array}$

$$\left(\begin{array}{l} 20 + .0 + .08 \\ \underline{7 + .3 + .09} \\ 27 + .3 + .17 = \\ 27 + .3 + .1 + .07 = \\ 27 + .4 + .07 = 27.47 \end{array}\right)$$

(5) $\begin{array}{r} 0.87 \\ \underline{+ 0.76} \end{array}$

$$\left(\begin{array}{l} .8 + .07 \\ \underline{.7 + .06} \\ 1.5 + .13 = \\ 1.5 + .1 + .03 = 1.63 \end{array}\right)$$

Our method can be shown in the vertical form you used for whole numbers:

$$\begin{array}{r} 14.56 \\ \underline{27.25} \\ .11 \\ .7 \\ 11. \\ \underline{30.} \\ 41.81 \end{array}$$

(Why did we mark the decimal point?)
(To show place value)

Of course this can be shortened by "remembering."

Here are the steps:

$$\begin{array}{r} 14.56 \\ \underline{27.25} \end{array}$$

(1) Add hundredths. Write 1 hundredth, remember 1 tenth.

$$\begin{array}{r} 14.56 \\ \underline{27.25} \\ 1 \end{array}$$

(2) Add tenths. Write 8 tenths. *Mark the decimal point.*

$$\begin{array}{r} 14.56 \\ \underline{27.25} \\ .81 \end{array}$$

(3) Add ones. Write 1 one and remember 1 ten.

$$\begin{array}{r} 14.56 \\ \underline{27.25} \\ 1.81 \end{array}$$

(4) Add tens. Write 4 tens.

$$\begin{array}{r} 14.56 \\ \underline{27.25} \\ 41.81 \end{array}$$

Here is an example with more remembering. Only the long way is shown:

$$\begin{array}{r} 17.67 \\ \underline{+\ 8.34} \\ .11 \\ .9 \\ 15. \\ \underline{10.} \\ 26.01 \end{array}$$

Exercise Set 29

Find fraction or decimal names for the following sums. The forms suggested below may be used.

$$\begin{aligned} 2\tfrac{3}{8} + 3\tfrac{2}{8} &= (2 + \tfrac{3}{8}) + (3 + \tfrac{2}{8}) \\ &= (2 + 3) + (\tfrac{3}{8} + \tfrac{2}{8}) \\ &= 5 + \tfrac{5}{8} \\ &= 5\tfrac{5}{8} \end{aligned}$$

$$\begin{aligned} 4.5 + 3.3 &= (4 + .5) + (3 + .3) \\ &= (4 + 3) + (.5 + .3) \\ &= 7 + .8 \\ &= 7.8 \end{aligned}$$

1. (a) $3\frac{3}{5} + 4\frac{1}{5}$ ($7\frac{4}{5}$) (b) $8\frac{4}{8} + 3\frac{3}{8}$ ($11\frac{7}{8}$) (c) $2\frac{2}{4} + 1\frac{1}{4}$ ($3\frac{3}{4}$)

2. (a) $1\frac{1}{6} + 6\frac{4}{6}$ ($7\frac{5}{6}$) (b) $8\frac{5}{12} + 6\frac{6}{12}$ ($14\frac{11}{12}$) (c) $4\frac{5}{16} + 5\frac{9}{16}$ ($9\frac{7}{8}$)

3. (a) 2.5 + 4.2 (6.7) (b) 8.6 + 3.3 (11.9) (c) 5.4 + 8.5 (13.9)

4. (a) 5.3 + 6.6 (11.9) (b) 8.4 + 4.4 (12.8) (c) 7.7 + 3.2 (10.9)

Try to work these mentally. Write the answers only.

5. (a) $8\frac{3}{5} + 5$ $(13\frac{3}{5})$ (b) $7 + 6\frac{2}{8}$ $(13\frac{1}{4})$ (c) $8 + 5\frac{5}{12}$ $(13\frac{5}{12})$

6. (a) $8.6 + 4.2$ (12.8) (b) $9.7 + 6.2$ (15.9) (c) $8.3 + 3.4$ (11.7)

BRAINTWISTER

7. Find n so each mathematical sentence is true.

(a) $(4\frac{1}{6} + 3\frac{2}{6}) + 5\frac{2}{6} = n$ $(12\frac{5}{6})$

(b) $(8.7 + 6.9) + 3.5 = n$ (19.1)

(c) $(6\frac{2}{8} + \frac{3}{8}) + n = 10\frac{7}{8}$ $(4\frac{1}{4})$

(d) $(11.2 + 8.7) + 1.8 = n$ (21.7)

Exercise Set 30

Copy and find the sums. The form of exercises 1(a) and 1(b) may be used.

1. (a)
$$\begin{array}{r} 2\frac{3}{10} = 2 + \frac{3}{10} \\ \underline{5\frac{8}{10}} = \underline{5 + \frac{8}{10}} \\ 7 + \frac{11}{10} = 8\frac{1}{10} \end{array}$$

(b)
$$\begin{array}{r} 2.3 = 2 + .3 \\ \underline{5.8} = \underline{5 + .8} \\ 7 + 1.1 = 8.1 \end{array}$$

2. (a) $\begin{array}{r} 5\frac{3}{8} \\ \underline{4\frac{7}{8}} \\ (10\frac{1}{4}) \end{array}$ (b) $\begin{array}{r} 4\frac{5}{6} \\ \underline{3\frac{4}{6}} \\ (8\frac{1}{2}) \end{array}$ (c) $\begin{array}{r} 5\frac{3}{4} \\ \underline{4\frac{7}{8}} \\ (10\frac{5}{8}) \end{array}$ (d) $\begin{array}{r} 7\frac{2}{4} \\ \underline{6\frac{3}{4}} \\ (14\frac{1}{4}) \end{array}$

3. (a) $\begin{array}{r} 5.8 \\ \underline{6.3} \\ (12.1) \end{array}$ (b) $\begin{array}{r} 6.4 \\ \underline{8.9} \\ (15.3) \end{array}$ (c) $\begin{array}{r} 8.7 \\ \underline{9.9} \\ (18.6) \end{array}$ (d) $\begin{array}{r} 8.5 \\ \underline{6.8} \\ (15.3) \end{array}$

4. (a) $\begin{array}{r} 8.6 \\ \underline{9.5} \\ (18.1) \end{array}$ (b) $\begin{array}{r} 6\frac{3}{5} \\ \underline{7\frac{4}{5}} \\ (14\frac{2}{5}) \end{array}$ (c) $\begin{array}{r} 8\frac{7}{12} \\ \underline{6\frac{10}{12}} \\ (15\frac{5}{12}) \end{array}$ (d) $\begin{array}{r} 9.8 \\ \underline{8.9} \\ (18.7) \end{array}$

Copy these examples and add. Write only the answers on your paper.

5. (a) $\begin{array}{r} 4.5 \\ \underline{3.3} \\ (7.8) \end{array}$ (b) $\begin{array}{r} 5.7 \\ \underline{4.2} \\ (9.9) \end{array}$ (c) $\begin{array}{r} 8.3 \\ \underline{9.5} \\ (17.8) \end{array}$ (d) $\begin{array}{r} 5.6 \\ \underline{4.9} \\ (10.5) \end{array}$

6. (a) $\begin{array}{r} 4.7 \\ \underline{4.6} \\ (9.3) \end{array}$ (b) $\begin{array}{r} 5.4 \\ \underline{6.9} \\ (12.3) \end{array}$ (c) $\begin{array}{r} 7.6 \\ \underline{4.8} \\ (12.4) \end{array}$ (d) $\begin{array}{r} 4.5 \\ \underline{2.7} \\ (7.2) \end{array}$

Exercise Set 31

1. Compute:

a.
$$\begin{array}{r} 25.06 \\ +\ 37.84 \\ \hline (62.90) \end{array}$$

b.
$$\begin{array}{r} 108.07 \\ +\ 467.94 \\ \hline (576.01) \end{array}$$

c.
$$\begin{array}{r} 117.6 \\ +\ 38.74 \\ \hline (156.34) \end{array}$$

d.
$$\begin{array}{r} .58 \\ +\ 15.09 \\ \hline (15.67) \end{array}$$

e.
$$\begin{array}{r} .847 \\ +\ \ .138 \\ \hline (.985) \end{array}$$

f.
$$\begin{array}{r} 3.707 \\ +\ 2.988 \\ \hline (6.695) \end{array}$$

2. Compute the sum in any language:

Example: $4\frac{1}{2} + 6.7$

$$\begin{array}{r} 4.5 \\ +\ 6.7 \\ \hline 11.2 \end{array} \qquad \text{or} \qquad \begin{array}{r} 4\frac{1}{2} \\ 6\frac{7}{10} \\ \hline 10\frac{12}{10} = 11\frac{1}{5} \end{array}$$

a. $23\frac{1}{2} + \frac{7}{4}$ $(25\frac{1}{4})$

b. $15\frac{1}{4} + 16.7$ (31.95)

c. $\frac{3}{100} + 18.57$ (18.6)

d. $15\frac{7}{8} + 18\frac{1}{3}$ $(34\frac{5}{24})$

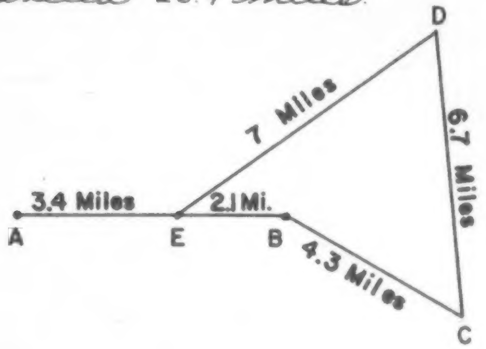
e. $6\frac{1}{4} + 7.18 + \frac{2}{5}$ (13.83)

The way we name money values in dollars is really a decimal numeral and symbol, the dollar sign, which indicates the unit.

$12.98 is usually read "twelve dollars and ninety-eight cents," but it could just as well be read "twelve and ninety-eight one hundredths dollars."

3. Stores often sell things at prices like $1.98 or $.49. If you bought something for $2.98 and something for $1.69, could you pay for them with a 5 dollar bill?
($2.98 + 1.69 = n$; $n = 4.67$. $4.67 < 5$. You could pay for them with a 5 dollar bill.)

4. Here is a map showing a short trip Ellen's family took in a car. They went from A (home) to B to C to D to E, and back to A. How far did they travel?
($3.4 + 2.1 + 4.3 + 6.7 + 7 + 3.4 = n$; $n = 26.9$. They traveled 26.9 miles.)



SUBTRACTION OF RATIONAL NUMBERS USING DECIMALS

Objective: To use understanding of place value in the decimal system to develop computational procedures for subtraction of rational numbers.

Materials: Pocket Chart and cards.

Suggested Teaching Procedure

The chief difficulty pupils encounter in subtracting rational numbers using decimal numerals is in the regrouping necessary when a digit in the addend is greater than the corresponding digit in the sum. It is advisable to have the pupils use expanded notation when they encounter this difficulty, and also to make use of the pocket chart and cards to illustrate the regrouping required. Pupils should also be encouraged to record regrouping as shown on the second page of the Exploration when they are having difficulty in computing correctly without this help.

SUBTRACTION OF RATIONAL NUMBERS USING DECIMALS

Exploration

Do you remember how you found a process for subtracting whole numbers using decimal notation? (Recall that "decimal" means "base ten.")

To get the decimal numeral for

$$237 - 145$$

you thought this way.

$$\begin{array}{rcl} 237 & = & 200 + 30 + 7 \\ -\ 145 & = & \underline{100 + 40 + 5} \\ & & ? + ? + 2 \end{array}$$

Then you thought

$$\begin{array}{rcl} 237 & = & 100 + 130 + 7 \\ -\ 145 & = & \underline{100 + \ 40 + 5} \\ & & \qquad 90 + 2 = 92 \end{array}$$

Can we think this way if our problem is

$$2.37 - 1.45 = ?$$

$$\begin{array}{rcl} 2.37 & = & 2 + .3 + .07 = 1 + 1.3 + .07 \\ 1.45 & = & 1 + .4 + .05 = \underline{1 + \ .4 + .05} \\ & & \qquad\qquad\qquad .9 + .02 = .92 \end{array}$$

Here we thought of 1.3 as 13 tenths. 13 tenths $-$ 4 tenths $=$ 9 tenths. We see that the idea is exactly the same. Here is one more example:

$$3.08 - 1.9 = ?$$

$$\begin{array}{rcl} 3.08 & = & 3 + .0 + .08 = 2 + 1.0 + .08 \\ 1.9 & = & 1 + .9 + .00 = \underline{1 + \ .9 + .00} \\ & & \qquad\qquad\qquad 1 + \ .1 + .08 = 1.18 \end{array}$$

Here it helped us to think of $.08$ as $(.0 + .08)$ and $.9$ as $(.9 + .00)$.

Try these examples. Write your work as shown on the preceding page.

1) 9.25 - 4.13 (5.12) 2) 18.36 - 2.5 (15.86) 3) 8.46 - 3.59 (4.87)

We can shorten this method if we think but do not write all of the steps. Here is one way:

$$\begin{array}{r} 3.08 \\ -\,1.9 \\ \hline \end{array}$$

a) Write this as:

$$\begin{array}{r} 3.08 \\ -\,1.90 \\ \hline \end{array}$$

b) Subtract hundredths:

$$\begin{array}{r} 3.08 \\ -\,1.90 \\ \hline 8 \end{array}$$

c) Think 2 + 1.0 for 3.0. Write this in tenths if you need to:

$$\begin{array}{r} \overset{\textcircled{2}}{\cancel{3}}.\overset{\textcircled{10}}{\cancel{0}}8 \\ -\,1.90 \\ \hline 8 \end{array}$$

d) Subtract tenths. 10 tenths - 9 tenths = 1 tenth. Mark the decimal point.

$$\begin{array}{r} \overset{\textcircled{2}}{\cancel{3}}.\overset{\textcircled{10}}{\cancel{0}}8 \\ -\,1.90 \\ \hline .18 \end{array}$$

e) Subtract ones:

$$\begin{array}{r} \overset{\textcircled{2}}{\cancel{3}}.\overset{\textcircled{10}}{\cancel{0}}8 \\ -\,1.90 \\ \hline 1.18 \end{array}$$

Try these examples the short way.

4) 7.38 - 5.2 (2.18) 5) 12.49 - 8.62 (3.87) 6) 10.37 - 4.59 (5.78)

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Exercise Set 32

Copy and subtract. Exercise 1(a) is done for you.

1. (a) 5.6 = 4 + 16 tenths
3.7 = 3 + 7 tenths
1 + 9 tenths = 1.9

	(b)	(c)
	4.7	5.8
	1.9	2.9
	(2.8)	(2.9)

	(a)	(b)	(c)	(d)
2.	5.6	8.1	7.6	8.5
	2.9	4.7	3.7	2.9
	(2.7)	(3.4)	(3.9)	(5.6)

Try to do these subtractions mentally. Write only the results for exercises 3 through 5.

	(a)	(b)	(c)	(d)
3.	7.1	8.7	7.4	8.3
	3.9	2.9	3.5	4.4
	(3.2)	(5.8)	(3.9)	(3.9)
4.	8.2	9.0	18.2	9.3
	3.4	3.6	6.5	5.8
	(4.8)	(5.4)	(11.7)	(3.5)
5	9.8	8.6	7.5	8.3
	7.9	6.8	3.8	5.6
	(1.9)	(1.8)	(3.7)	(2.7)

6. BRAINTWISTER. Fill in the squares so the sum of each row and column is the same number.

(a)

2.4	(5.4)	1.2
(1.8)	3.0	(4.2)
4.8	(.6)	3.6

(b)

.8	.1	(.6)
(.3)	.5	(.7)
(.4)	.9	(.2)

Exercise Set 33

Copy and subtract. Exercise 1(a) is done for you.

1. (a) $7.83 = 7 + .7 + .13$

$5.35 = \underline{5 + .3 + .05}$

$2 + .4 + .08 = 2.48$

(b) 2.48
1.09
(1.37)

2.	(a) 6.85 2.49 (4.36)	(b) 7.74 3.37 (4.37)	(c) 9.96 4.37 (5.59)	(d) 8.86 3.57 (5.29)
3.	(a) 7.61 3.36 (4.25)	(b) 8.94 2.78 (6.16)	(c) 5.50 4.37 (1.13)	(d) 9.72 3.69 (6.03)

Subtract these mentally. Write just the answers.

4.	(a) 7.34 3.28 (4.06)	(b) 8.92 3.47 (5.45)	(c) 9.71 3.58 (6.13)	(d) 8.54 6.39 (2.15)
5.	(a) 9.65 3.39 (6.26)	(b) 8.47 4.38 (4.09)	(c) 9.83 4.56 (5.27)	(d) 7.81 4.64 (3.17)

6. Subtract

(a) 8.34 4.83 (3.51)	(b) 9.28 7.85 (1.43)	(c) 8.32 4.58 (3.74)	(d) 9.34 5.89 (3.45)

BRAINTWISTER

7. Find n so each of the following is a true mathematical sentence.

(a) $\frac{5}{6 \times 2} + \frac{8}{4 \times 3} = n$ $(1\frac{1}{12})$ (c) $\frac{4}{2 \times 8} + \frac{7}{4 + 4} = n$ $(1\frac{1}{8})$

(b) $\frac{4 + 3}{4 + 1} + \frac{3}{3 + 2} = n$ (2) (d) $\frac{5 \times 3}{8 + 2} + \frac{3 + 4}{5 \times 5} = n$ (1.78)

Exercise Set 34

You have learned these two methods to express $2.52 + 5.46$ as a decimal numeral

$$\begin{array}{rl} 2.52 = & 2\frac{52}{100} \\ 5.46 = & \underline{5\frac{46}{100}} \\ & 7\frac{98}{100} = 7.98 \end{array}$$

$$\begin{array}{rl} 2.52 = & 2 + .5 + .02 \\ 5.46 = & \underline{5 + .4 + .06} \\ & 7 + .9 + .08 = 7.98 \end{array}$$

You have learned these two methods to express $5.84 - 3.32$ as a decimal numeral.

$$\begin{array}{rl} 5.84 = & 5\frac{84}{100} \\ 3.32 = & \underline{3\frac{32}{100}} \\ & 2\frac{52}{100} = 2.52 \end{array}$$

$$\begin{array}{rl} 5.84 = & 5 + .8 + .04 \\ 3.32 = & \underline{3 + .3 + .02} \\ & 2 + .5 + .02 = 2.52 \end{array}$$

Use two methods to add in exercise 1.

1. (a) $\begin{array}{r} 5.63 \\ \underline{+\ 2.34} \\ (7.97) \end{array}$ (b) $\begin{array}{r} 6.35 \\ \underline{+\ 3.44} \\ (9.79) \end{array}$ (c) $\begin{array}{r} 7.24 \\ \underline{+\ 8.33} \\ (15.57) \end{array}$ (d) $\begin{array}{r} 5.56 \\ \underline{+\ 3.33} \\ (8.89) \end{array}$

Use two methods to subtract in exercise 2.

2.	(a) 6.24	(b) 8.69	(c) 7.87	(d) 8.86
	- 3.12	- 4.35	- 5.36	- 4.24
	(3.12)	(4.34)	(2.51)	(4.62)

Write only your answers for exercises 3 and 4. Do your work mentally.

3.	(a) 4.62	(b) 8.36	(c) 5.21	(d) 6.54
	+ 3.26	+ 3.43	+ 3.47	+ 3.35
	(7.88)	(11.79)	(8.68)	(9.89)
4.	(a) 6.57	(b) 8.79	(c) 9.68	(d) 8.89
	- 3.35	- 5.34	- 4.35	- 3.67
	(3.22)	(3.45)	(5.33)	(5.22)

5. BRAINTWISTER: Find n so each mathematical sentence is true.

(a) (8.97 - 4.31) + n = 11.89 (7.23)

(b) 11.89 - n = 8.97 - 4.31 (7.23)

Exercise Set 35

You have shown your work for renaming 6.28 + 3.57 as a decimal numeral in two ways.

$$\begin{aligned} 6.28 &= 6 + .2 + .08 \\ \underline{3.57} &= \underline{3 + .5 + .07} \\ &= 9 + .7 + .15 \\ &= 9 + .7 + .1 + .05 \\ &= 9.85 \end{aligned}$$

$$\begin{array}{r} 6.28 \\ \underline{3.57} \\ .15 \\ .70 \\ \underline{9.00} \\ 9.85 \end{array}$$

Use both methods to find the sums in exercise 1.

1.

(a)	(b)	(c)	(d)
3.49	5.64	6.28	6.29
+ 2.38	+ 3.18	+ 3.47	+ 4.38
(5.87)	(8.82)	(9.75)	(10.67)

Write only the sums for exercises 2 and 3.

2.

(a)	(b)	(c)	(d)
6.29	3.42	7.23	8.25
5.38	4.48	3.58	3.37
(11.67)	(7.90)	(10.81)	(11.62)

3.

(a)	(b)	(c)	(d)
7.34	8.23	9.34	8.38
5.37	4.39	2.59	5.52
(12.71)	(12.62)	(11.93)	(13.90)

4. BRAINTWISTERS: Find these sums. Write only the decimal numeral for the sum.

(a)	(b)	(c)	(d)
3.24	4.46	5.36	7.88
3.56	3.32	4.75	5.31
4.16	5.51	2.83	6.54
(10.96)	(13.29)	(12.94)	(19.73)

Exercise Set 36

1. Compute. Show your work the long way.

(a) 15.27 - 4.81 (10.46) (b) 3.75 - 0.28 (3.47) (c) 28.75 - 13.86 (14.89)

2. Compute:

(a) 86.23
- 57.70
(28.53)

(b) 862.3
- 57.7
804.6

(c) 90 - 67.86 (22.14)

(d) 50.09
- 4.91
(45.18)

(e) .125
+ .375
(.500)

(f) 75 - 7.83 (67.17)

(g) Translate each addend in (e) to a fraction in simplest form. Compute in fractions. $\left(\frac{1}{8}+\frac{3}{8}=\frac{4}{8}=\frac{1}{2}\right)$

3. Compute these repeated sums. Use the long way if you cannot remember.

(a) 7.08
38.92
+ 16.60
(62.60)

(b) .73
38.
+ 6.94
(45.67)

(c) 16.67
3.04
.08
+ 6.20
(25.99)

4. How much change should you get from a 10 dollar bill if you bought things costing \$3.98, \$1.49, \$.98, and \$1.69? ($(10 - (3.98 + 1.49 + .98 + 1.69) = n$; $n = 1.86$. You should get one dollar and 86 cents in change)

5. The population of the United States was 131.669 million in 1940. 150.697 million in 1950 and 179.323 million in 1960. Did the pupulation increase more between 1940 and 1950 or between 1950 and 1960?
($150.697 - 131.669 = n$; $n = 19.028$.
$179.323 - 150.697 = t$; $t = 28.626$.
The population increased more between 1950 and 1960.)

6. Here are the populations of the 5 largest cities in the United States in 1960. Was the part of the United States population which lives in these cities more or less than $\frac{1}{10}$ of the total population? ($7.78 + 3.55 + 2.48 + 2.00 + 1.67 = n$; $n = 17.48$
$17.48 < \frac{1}{10} \times 179.323$. The population living in these cities was less than $\frac{1}{10}$ of the total population.)

New York	7.78	million
Chicago	3.55	million
Los Angeles	2.48	million
Philadelphia	2.00	million
Detroit	1.67	million

REVIEW

Objective: To help pupils review the meanings, skills, and procedures for problem solving developed in this unit.

Exploration:

Three exercise sets are included in this section. Sets 38 and 39 consist of word problems. Pupils should write mathematical sentences and use methods of solving problems described earlier in this commentary. The last part, called Just For Fun, is a magic square.

Not all pupils need solve every exercise or problem. The large number of exercises and problems permit making assignments suitable to the ability of each pupil. At the same time, these sets are not completely review. There are many variations of the content studied previously. There should be some class discussion of the difficult exercises and methods for attacking them.

More capable pupils may work on Exercise Set 40 and Just For Fun when they have satisfactorily completed this section.

REVIEW

Exercise Set 37

1. What number n will make each mathematical sentence true?

(a) $\frac{1}{4} = \frac{n}{8}$ (2) (c) $\frac{3}{4} = \frac{n}{8}$ (6) (e) $\frac{7}{8} = \frac{14}{n}$ (16)

(b) $\frac{1}{2} = \frac{n}{12}$ (6) (d) $\frac{n}{4} = \frac{12}{16}$ (3) (f) $\frac{5}{n} = \frac{10}{16}$ (8)

2. Copy and write " > ", " < ", or " = " in each blank so each mathematical sentence is true.

(a) $\frac{1}{3}$ (>) $\frac{1}{8}$ (c) $\frac{1}{9}$ (<) $\frac{1}{5}$ (e) $\frac{2}{3}$ (<) $\frac{3}{4}$

(b) $\frac{1}{7}$ (<) $\frac{1}{5}$ (d) $\frac{2}{3}$ (=) $\frac{12}{18}$ (f) $\frac{9}{8}$ (<) $\frac{4}{3}$

3. Which would you rather have?

(a) $\frac{1}{4}$ or $\frac{1}{8}$ of a pie $(\frac{1}{4})$ (c) $\frac{1}{4}$ or $\frac{1}{3}$ of a candy bar $(\frac{1}{3})$

(b) $\frac{1}{4}$ or $\frac{1}{5}$ of a dollar $(\frac{1}{4})$ (d) $\frac{1}{8}$ or $\frac{1}{6}$ of a watermelon $(\frac{1}{6})$

4. Find a fraction name for n so each mathematical sentence is true.

(a) $\frac{1}{2} + n = \frac{5}{8}$ $(\frac{1}{8})$ (c) $\frac{1}{4} + n = \frac{3}{12} + \frac{7}{12}$ $(\frac{7}{12})$

(b) $\frac{1}{2} + n = \frac{3}{6} + \frac{2}{6}$ $(\frac{1}{3})$ (d) $n + \frac{1}{2} = \frac{3}{6} + \frac{4}{6}$ $(\frac{2}{3})$

5. Which fractions name numbers greater than the number 1?

$\frac{1}{4}, \frac{7}{4}, \frac{3}{8}, \frac{19}{16}, \frac{9}{10}, \frac{12}{10}, \frac{15}{9}, \frac{8}{9}, \frac{4}{6}, \frac{6}{5}$

$(\frac{7}{4}, \frac{19}{16}, \frac{12}{10}, \frac{15}{9}, \frac{6}{5})$

6. Find the simplest fraction name for each number.

(a) $\frac{16}{24}$ $(\frac{2}{3})$ (c) $\frac{18}{24}$ $(\frac{3}{4})$ (e) $\frac{4}{20}$ $(\frac{1}{5})$ (g) $\frac{14}{18}$ $(\frac{7}{9})$ (i) $\frac{14}{21}$ $(\frac{2}{3})$

(b) $\frac{10}{20}$ $(\frac{1}{2})$ (d) $\frac{12}{15}$ $(\frac{4}{5})$ (f) $\frac{16}{20}$ $(\frac{4}{5})$ (h) $\frac{24}{30}$ $(\frac{4}{5})$ (j) $\frac{10}{16}$ $(\frac{5}{8})$

7. Rename each number in mixed form.

(a) $\frac{16}{9}$ $(1\frac{7}{9})$ (c) $\frac{8}{3}$ $(2\frac{2}{3})$ (e) $\frac{17}{12}$ $(1\frac{5}{12})$

(b) $\frac{12}{7}$ $(1\frac{5}{7})$ (d) $\frac{19}{5}$ $(3\frac{4}{5})$ (f) $\frac{13}{5}$ $(2\frac{3}{5})$

8. Copy each statement below. For each missing numerator or denominator write a numeral so each mathematical sentence is true.

(a) $\frac{8}{12} = \frac{(2)}{3} = \frac{(4)}{6} = \frac{(12)}{18} = \frac{(16)}{24}$

(b) $\frac{9}{15} = \frac{(12)}{20} = \frac{(21)}{35} = \frac{6}{(10)} = \frac{15}{(25)}$

(c) $\frac{3}{18} = \frac{(2)}{12} = \frac{4}{(24)} = \frac{5}{(30)} = \frac{(6)}{36}$

9. Which of these fractions are other names for $\frac{1}{2}$?

$\frac{6}{18}$ $\frac{10}{25}$ $\frac{4}{5}$ [✓ $\frac{5}{10}$] $\frac{7}{5}$ $\frac{8}{6}$ [✓ $\frac{2}{4}$]

$\frac{7}{8}$ [✓ $\frac{8}{16}$] [✓ $\frac{6}{12}$] $\frac{13}{21}$ $\frac{6}{4}$ $\frac{3}{5}$ [✓ $\frac{4}{8}$]

[✓ $\frac{14}{28}$] [✓ $\frac{3}{6}$] $\frac{8}{5}$ [✓ $\frac{7}{14}$] [✓ $\frac{9}{18}$] $\frac{5}{7}$ $\frac{12}{8}$

10. Which fractions in exercise 9 are other names for $\frac{9}{6}$? $(\frac{6}{4}$ and $\frac{12}{8})$

11. Which pairs of numbers below can be named by fractions with a common denominator of 24 ? (a, d, e)

(a) $\frac{1}{2}$ and $\frac{4}{3}$? (d) $\frac{10}{3}$ and $\frac{1}{8}$?

(b) $\frac{4}{6}$ and $\frac{4}{7}$? (e) $\frac{7}{6}$ and $\frac{3}{8}$?

(c) $\frac{7}{4}$ and $\frac{4}{7}$? (f) $\frac{5}{2}$ and $\frac{7}{9}$?

Exercise Set 38

1. There were $2\frac{2}{3}$ qt. vanilla ice cream and $3\frac{1}{2}$ qt. chocolate ice cream in the freezer. How much ice cream was in the freezer? ($2\frac{2}{3} + 3\frac{1}{2} = n$; $n = 6\frac{1}{6}$. There were $6\frac{1}{6}$ quarts of ice cream in the freezer.)

2. Mary's mother made two costumes for a school play. One costume took $3\frac{2}{3}$ yards material and the other costume took $2\frac{1}{6}$ yards. How much material did Mary's mother buy? ($3\frac{2}{3} + 2\frac{1}{6} = n$; $n = 5\frac{5}{6}$. She bought $5\frac{5}{6}$ yards.)

3. Dick's weight was $56\frac{3}{4}$ lb. in June. At the end of vacation he weighed $59\frac{1}{2}$ lb. How much weight did he gain? ($56\frac{3}{4} + n = 59\frac{1}{2}$; $n = 2\frac{3}{4}$. He gained $2\frac{3}{4}$ pounds.)

4. Mr. Long noticed that the odometer in his car showed 8523.4 miles when he bought some gas. During the day he traveled 49.3 miles. What did his odometer read at the end of the day? ($8523.4 + 49.3 = n$; $n = 8572.7$. The odometer read 8572.7.)

5. Bruce wanted to buy a sleeping bag for a Scout camping trip that was priced \$26.95. Bruce's father gave him \$5 to help pay the cost. Bruce had saved \$7.35. How much more money does Bruce need to buy the sleeping bag? ($5 + 7.35 + n = 26.95$; $n = 14.60$. Bruce needed \$14.60 more.)

6. Gerry was ill one day and her mother took her temperature in the morning. The thermometer read 99.8°. Later in the day Gerry's fever increased and the thermometer read 102.6°. How many degrees did her fever increase? ($99.8 + n = 102.6$; $n = 2.8$. Her fever increased 2.8 degrees.)

7. Tell whether each of these mathematical sentences is an example of the commutative or the associative properties for addition.

(a) $\frac{2}{3} + (\frac{1}{4} + \frac{3}{4}) = (\frac{2}{3} + \frac{1}{4}) + \frac{3}{4}$ (associative)

(b) $\frac{3}{12} + \frac{2}{8} = \frac{2}{8} + \frac{3}{12}$ (commutative)

(c) $(\frac{8}{9} + \frac{12}{3}) + \frac{1}{6} = \frac{8}{9} + (\frac{12}{3} + \frac{1}{6})$ (associative)

(d) $(\frac{1}{3} + \frac{1}{4}) + \frac{2}{3} = (\frac{1}{4} + \frac{1}{3}) + \frac{2}{3}$ (commutative)

(e) $(\frac{8}{9} + \frac{4}{5}) + \frac{1}{2} = \frac{8}{9} + (\frac{4}{5} + \frac{1}{2})$ (associative)

8. In his butcher shop, Mr. Fisher had some bologna in chunks. On Monday he sold $2\frac{7}{8}$ lb. The next day he sold $\frac{3}{4}$ lb. On Wednesday he sold $3\frac{1}{2}$ lb.

Use the above information to complete problems (a) through (d).

(a) How many pounds of bologna did Mr. Fisher sell in the three days? ($2\frac{7}{8} + \frac{3}{4} + 3\frac{1}{2} = n$; $n = 7\frac{1}{8}$. He sold $7\frac{1}{8}$ pounds.)

(b) How much less than 10 lb. was sold? ($10 - 7\frac{1}{8} = n$; $n = 2\frac{7}{8}$. He sold $2\frac{7}{8}$ lbs. less than 10 lbs.)

(c) How much bologna was sold on Tuesday and Wednesday? ($\frac{3}{4} + 3\frac{1}{2} = n$; $n = 4\frac{1}{4}$. He sold $4\frac{1}{4}$ lbs. on Tuesday and Wednesday.)

(d) The total number pounds of bologna sold on the last two days is how many more than the number of pounds sold on the first two days? Express your answer in simplest form. ($2\frac{7}{8} + \frac{3}{4} = n$; $n = 3\frac{5}{8}$. $\frac{3}{4} + 3\frac{1}{2} = t$; $t = 4\frac{1}{4}$. $4\frac{1}{4} - 3\frac{5}{8} = p$; $p = \frac{5}{8}$. Or, $3\frac{1}{2} - 2\frac{7}{8} = k$; $k = \frac{5}{8}$. The total number of pounds sold on the last two days was $\frac{5}{8}$ lbs. more than the number of pounds sold on the first two days.)

Exercise Set 39

1. The fastest pitched ball on record traveled 98.6 m.p.h. When a hockey player strikes a puck, the puck travels about 98.0 m.p.h. Which traveled faster? How much? ($98.6 - 98.0 = n$; $n = .6$. The ball traveled .6 per mile faster.)

2. A recent census showed that out of every 100 people in South Carolina 36.7 lived in towns and cities, and 63.3 lived in rural communities. Out of every 100 people, how many more lived in rural communities? ($63.3 - 36.7 = n$; $n = 26.6$. 26.6 more lived in rural communities.)

3. The flight time of Explorer III was 115.87 minutes. That of Explorer I was 114.8. What is the difference in the two flight times? ($115.87 - 114.8 = n$; $n = 1.07$. The difference in the two flight times is 1.07 minutes.)

4. In George Washington's time, .90 of the American people could not read or write. Today only about .05 of the American people cannot read and write. What part of the people could read and write in Washington's time? In our time? ($1.00 - .90 = n$; $n = .10$. .10 of the American people could read and write in George Washington's time. $1.00 - .05 = n$; $n = .95$. .95 of the American people can read and write today.)

5. The Simplon Tunnel between Italy and Switzerland is 12.3 miles long. The Cascade Tunnel in Washington is 7.8 miles long. How much longer is the Simplon Tunnel? ($12.3 - 7.8 = n$; $n = 4.5$. The Simplon Tunnel is 4.5 miles longer.)

6. In 1950, statistics showed that the population per square mile in California was 66.7 persons. In 1940 it had been 43.7 persons. On the average, how many more people lived on a square mile in 1950? ($66.7 - 43.7 = n$; $n = 23$. On the average, 23 more people lived on a square mile in 1950.)

7. The Moosehead Lake in Maine has an area of 116.98 square miles. The area of Lake Mead in Nevada is 228.83 square miles. Which lake has the greater area? (Lake Mead) How much greater? ($228.83 - 116.98 = n$; $n = 111.85$. Lake Mead has an area 111.85 greater than Moosehead Lake.)

8. The length of a day on Mars is 24.5 hours. The length of a day on Neptune is 15.7 hours. How much longer is a day on Mars? ($24.5 - 15.7 = n$; $n = 8.8$. The day on Mars is 8.8 hours longer than a day on Neptune.)

9. The distance from Earth to Cygnus is 10.6 light years. The distance from Earth to Sirius is 8.6 light years. Which star is closer to Earth? How much closer? ($10.6 - 8.6 = n$; $n = 2$. Sirius is 2 light years closer to Earth.)

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JUST FOR FUN

Make a square like this.

1 (8)	2 (3.5)	3 (3.5)	4 (2)
5 (1)	6 (5.5)	7 (3.50)	8 (7)
9 (1.50)	10 (5)	11 (3)	12 (7.50)
13 (6.5)	14 (3)	15 (7)	16 (.5)

In each small square, write the answer of the example below having the same numeral as the square. If your work is correct, the sum of the numbers of each row and column will be the same number.

1. 6.50 + 1.50
2. 4.75 - 1.25
3. 1.8 + 1.7
4. 96 ÷ 48
5. .25 + .75
6. 8.00 - 2.50
7. 1.75 + 1.75
8. 63 ÷ 9
9. .70 + .80
10. 4.85 + .15
11. 5.37 - 2.37
12. 6 + 1.50
13. 7.7 - 1.2
14. 108 ÷ 36
15. 13.76 - 6.76
16. 7.25 - 6.75

Exercise Set 40

1. Below are a number of steps showing $\frac{1}{2} + \frac{2}{3} = \frac{2}{3} + \frac{1}{2}$

State a reason for each step. Let $n = \frac{1}{2} + \frac{2}{3}$.

Then $n = \frac{3}{6} + \frac{4}{6}$ Step 1: ($\frac{1}{2}$ renamed $\frac{3}{6}$; $\frac{2}{3}$ renamed $\frac{4}{6}$.)

$= \frac{3 + 4}{6}$ Step 2: (addition with fraction numerals)

$= \frac{4 + 3}{6}$ Step 3: (Commutative property of addition)

$= \frac{4}{6} + \frac{3}{6}$ Step 4: (addition with fractions)

$= \frac{2}{3} + \frac{1}{2}$ Step 5: ($\frac{4}{6}$ renamed $\frac{2}{3}$; $\frac{3}{6}$ renamed $\frac{1}{2}$.)

2. Write steps like those in exercise 1 to show that

$\frac{1}{3} + \frac{3}{4} = \frac{3}{4} + \frac{1}{3}$. (Same as exercise 1 only $\frac{1}{3}$ and $\frac{3}{4}$ are renamed $\frac{4}{12}$ and $\frac{9}{12}$)

3. Study this sentence: $(\frac{7}{4} - n) - 1 = \frac{7}{4} - (n + 1)$. Is this sentence true?

(a) if $n = \frac{3}{4}$? (yes) (b) if $n = \frac{1}{4}$? (yes) (c) if $n = \frac{0}{4}$? (yes)

4. In each sentence what number does n represent if the mathematical sentence is true? Be careful. There may be no answer, one answer or more than one answer.

(a) $\frac{3}{4} + n = \frac{5}{4}$ ($\frac{2}{4}$) (d) $n = n$ (any rational number) (g) $3.4 + n = n$ (no rational number)

(b) $\frac{7}{4} + n = \frac{21}{10}$ ($\frac{7}{20}$) (e) $n + \frac{5}{4} = \frac{5}{4} + n$ (any rational number) (h) $n = 9.17 + n$ (no rational number)

(c) $1\frac{2}{3} + n = 3\frac{1}{4}$ ($\frac{19}{12}$) (f) $11.6 + n = 11.6$ (0) (g) $6.432 + n = n$ (no rational number)

5. (a) What number is n if $n + n = \frac{14}{5}$? $(\frac{7}{5})$

   (b) What number is n if $n + n = 3.94$? (1.97)

6. Think of x, y and z as representing rational numbers. Suppose $x + y = z$.

   (a) If $x = \frac{15}{4}$ and $z = \frac{15}{4}$, what number is y? (0)

   (b) Can x be greater than z? Why? (No, there is no rational number to use for a result)

   (c) If $y = 8.94$ and $z = 8.94$, what number is x? (0)

7. For each of the sentences below, n is a fraction name for a rational number. Make n have a denominator of 2. Find n if each mathematical sentence is true.

   (a) $n < \frac{1}{2}$ $(\frac{0}{2})$

   (b) n is less than $\frac{9}{2}$ and greater than $\frac{7}{2}$ $(\frac{8}{2})$

   (c) n is greater than $\frac{13}{4}$ and less than $\frac{15}{4}$. $(\frac{7}{2})$

   (d) The sum of n and $\frac{7}{2}$ is less than $\frac{8}{2}$ $(\frac{0}{2})$

8. One mathematical sentence showing a relationship among numbers n, 7 and 12 is $n + 7 = 12$. Write two mathematical sentences showing different relationships among n, 1.75, and 4.25. Make them so that for each mathematical sentence, n represents a different number.
($n + 1.75 = 4.25$. $1.75 + 4.25 = n$.)

9. Two numbers to be subtracted are represented by n and n. (The two numbers are the same.) John said the result of the subtraction is $\frac{10}{3}$. Was John correct? (no) Why?
(n - n must be 0 because n + 0 = n.)

10. What rational number n will make each mathematical sentence true?

(a) $(\frac{3}{2} + \frac{7}{2}) + n = \frac{11}{2}$ $(\frac{1}{2})$ (d) $(1.35 + 6.47) + n = 8.92$ (1.10)

(b) $(\frac{3}{2} + \frac{7}{2}) + \frac{11}{2} = n$ $(\frac{21}{2})$ (e) $(1.35 + 6.47) + 8.92 = n$ (16.74).

(c) $n - (\frac{3}{2} + \frac{7}{2}) = \frac{11}{2}$ $(\frac{21}{2})$ (f) $n - (1.35 + 6.47) = 8.92$ (16.74)

11. Sometimes you have more than two numbers to add. You may be able to make the exercise simpler by changing the order of the addends. For example, think about

$$\frac{5}{3} + \frac{1}{4} + \frac{2}{3} + \frac{3}{4} + \frac{5}{3}$$

You may think $(\frac{5}{3} + \frac{2}{3} + \frac{5}{3}) + (\frac{1}{4} + \frac{3}{4}) = 4 + 1 = 5$.

Find these sums. Change the order of the addends if you think it will make the computation easier.
(These are probably best.)

(a) $2 + \frac{2}{3} + 3 + \frac{2}{3} + \frac{2}{3}$ $((2+3) + (\frac{2}{3} + \frac{2}{3} + \frac{2}{3}) = 7)$

(b) $\frac{1}{2} + \frac{1}{4} + \frac{7}{4} + \frac{3}{2} + \frac{4}{4}$ $((\frac{1}{2} + \frac{3}{2}) + (\frac{1}{4} + \frac{7}{4} + \frac{4}{4}) = 5)$

(c) $\frac{8}{5} + \frac{7}{6} + \frac{3}{5} + \frac{9}{5} + \frac{11}{6}$ $((\frac{8}{5} + \frac{3}{5} + \frac{9}{5}) + (\frac{7}{6} + \frac{11}{6}) = 7)$

Practice Exercises

I. Solve for n .

a) $54{,}982 + n = 80{,}000$ $(n = 25{,}018)$

b) $300{,}678 + 27{,}\ 492 = n$ $(n = 328{,}170)$

c) $658 \times 319 = n$ $(n = 209{,}902)$

d) $n \times 85 = 4{,}080$ $(n = 48)$

e) $36 \times n = 2{,}700$ $(n = 75)$

f) $2{,}340 \div n = 36$ $(n = 65)$

g) $6\frac{5}{8} + 7\frac{1}{8} = n$ $(n = 13\frac{6}{8}$ or $13\frac{3}{4})$

h) $n + 8\frac{1}{5} = 11\frac{3}{10}$ $(n = 3\frac{1}{10})$

i) $\frac{3}{4} + \frac{2}{3} + \frac{5}{6} = n$ $(n = \frac{27}{12}$ or $2\frac{1}{4})$

j) $185 \times 85 = n$ $(n = 15{,}725)$

k) $2\frac{3}{5} - \frac{4}{5} = n$ $(n = 1\frac{4}{5})$

l) $32 \times n = 5024$ $(n = 157)$

m) $2\frac{2}{5} + \frac{1}{3} + 1\frac{1}{5} = n$ $(n = 3\frac{14}{15})$

n) $77 \times 34618 = n$ $(n = 2{,}665{,}586)$

o) $2\frac{1}{6} - \frac{2}{3} = n$ $(n = 1\frac{6}{12}$ or $1\frac{1}{2})$

p) $4\frac{1}{3} + 3\frac{5}{6} + n = 10$ $(n = 1\frac{5}{6})$

q) $n - 8\frac{1}{2} = \frac{5}{12}$ $(n = 8\frac{11}{12})$

r) $n = 1\frac{9}{10} + 6\frac{2}{5} + 3\frac{1}{2}$ $(n = 10\frac{18}{10}$ or $11\frac{4}{5})$

s) $4{,}006 \times 78 = n$ $(n = 312{,}468)$

t) $\frac{7}{8} + n = 4\frac{1}{2}$ $(n = 3\frac{5}{8})$

II. Solve for n .

a) $12\frac{1}{4} - n = 8\frac{1}{2}$ $(n = 3\frac{3}{4})$

b) $\frac{5}{8} + \frac{1}{2} = \frac{1}{2} + n$ $(n = \frac{5}{8})$

c) $\frac{7}{8} - \frac{1}{2} = \frac{1}{4} + n$ $(n = \frac{1}{8})$

d) $\frac{4}{5} + \frac{3}{10} + \frac{2}{5} = n$ $(n = \frac{15}{10}$ or $1\frac{1}{2})$

e) $5\frac{7}{10} - n = 2\frac{2}{3}$ $(n = 3\frac{1}{30})$

f) $2.45 + .7 + 3.05 = n$ $(n = 6.20)$

g) $248.09 + n = 388.6$ $(n = 140.51)$

h) $\frac{0}{4} + \frac{3}{4} = \frac{3}{4} + n$ $(n = 0)$

i) $n + 1\frac{1}{3} = 2\frac{1}{2}$ $(n = 1\frac{1}{6})$

j) $3{,}354 \div n = 39$ $(n = 86)$

k) $747.314 - 288.405 = n$ $(n = 458.909)$

l) $\frac{3}{5} + \frac{1}{2} + n = 1\frac{1}{2}$ $(n = \frac{4}{10}$ or $\frac{2}{5})$

m) $954 \times 384 = n$ $(n = 366{,}336)$

n) $11\frac{3}{8} + n = 14\frac{1}{3}$ $(n = 2\frac{23}{24})$

o) $6{,}400 \div 80 = n$ $(n = 80)$

p) $52{,}871 \div n = 91$ $(n = 581)$

q) $n \times 53 = 3{,}498$ $(n = 66)$

r) $37{,}030 \div 46 = n$ $(n = 805)$

s) $48{,}369 \times 789 = n$ $(n = 38{,}163{,}141)$

t) $4\frac{1}{3} + 5\frac{1}{2} + 3\frac{1}{4} = n$ $(n = 12\frac{13}{12}$ or $13\frac{1}{12})$

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III. Add:

a)	b)	c)	d)	e)
904	28,796	$3\frac{1}{5}$	$2\frac{5}{6}$	$2\frac{1}{2}$
652	29	$5\frac{3}{5}$	$\frac{1}{6}$	4
2,909	8,583	$2\frac{2}{5}$	$9\frac{1}{6}$	$6\frac{2}{3}$
45	61,312	$\frac{4}{5}$	$3\frac{5}{6}$	$3\frac{1}{6}$
(4510)	(98,720)	($10\frac{10}{5}$ or 12)	($14\frac{12}{6}$ or 16)	($15\frac{8}{6}$ or $16\frac{1}{3}$)

Subtract:

f)	g)	h)	i)	j)
5,934	17,004	$11\frac{7}{10}$	$3\frac{1}{2}$	$26\frac{1}{3}$
2,046	5,280	$4\frac{3}{10}$	$1\frac{1}{3}$	$9\frac{5}{6}$
(3 888)	(11,724)	($7\frac{4}{10}$ or $7\frac{2}{5}$)	($2\frac{1}{6}$)	($16\frac{3}{6}$ or $16\frac{1}{2}$)

Multiply:

k)	l)	m)	n)	o)
508	369	348	957	5,836
67	26	58	39	47
(34,036)	(9,594)	(20,184)	(37,323)	(274,292)

Divide:

p) 18)426 (23 r 12)

q) 42)4,309 (102 r 25)

r) 41)2,352 (57 r 15)

s) 37)1,591 (43)

t) 18)1,674 (93)

IV. Find the sum:

a) 264,829 ; 78,080 ; 196,809 ; 19,998 (559,716)

b) 132,435 ; 412,754 ; 216 ; 734,646 (1,280,051)

c) $28\frac{1}{3}$; $8\frac{1}{2}$; $17\frac{1}{4}$; $6\frac{2}{3}$; 19 ($79\frac{3}{4}$)

d) 4,027.9 ; 617.26 ; 503.07 ; .8 (5,149.03)

e) $219\frac{5}{8}$; $1,726\frac{1}{4}$; $63\frac{2}{3}$; $109\frac{3}{8}$ ($2,118\frac{11}{12}$)

V. Subtract:

a) 678,543 - 254,745 (423,798)

b) 800,096 - 173,295 (626,801)

c) 128,791 - 37,782 (91,009)

d) $52,096\frac{3}{8} - 29,636\frac{3}{4}$ $(22,459\frac{5}{8})$

e) 212,983 - 31,006 (181,977)

VI. Multiply:

a) 27,465 x 697 (19,143,105)

b) 379,865 x 756 (287,177,940)

c) 36,492 x 489 (17,844,588)

d) 843,476 x 654 (551,633,304)

e) 81,918 x 248 (20,315,664)

VII. Divide:

a) 85,591 ÷ 95 (900 r. 91)

b) 34,997 ÷ 34 (1,029 r 11)

c) 87,600 ÷ 67 (1,307 r 31)

d) 801,356 ÷ 89 (9,004)

e) 457,267 ÷ 74 (6179 r 21)

Review

SET I

Part A

1. Complete each of the following to make it a true statement illustrating the distributive property.

Example: 12 × (20 + 15) + (12 × 20) + (12 × 15)

a) (40 + 5) × 22 = (40 × 22) + (5 × 22)

b) 154 ÷ 7 = (140 + 14) ÷ 7

c) 468 × 15 = (424 × 15) + (44 × 15)

d) 1,824 ÷ 10 = (1000 ÷ 10) + (800 ÷ 10) + (20 ÷ 10) + 4

e) 63 × 34 = (60 + 3) × (30 + 4)

2. Answer yes or no to the questions below.

a) Does (7 × 8) × 3 = 7 × (8 × 3)? (yes)

b) Does 3 × (9 × 5) = (3 × 9) × 5 ? (yes)

c) Does (36 ÷ 6) ÷ 3 = 36 ÷ (6 ÷ 3) ? (no)

d) Does 60 ÷ (30 + 2) = (60 ÷ 30) ÷ 2 ? (no)

e) Does $(\frac{5}{4} - \frac{2}{4}) - \frac{1}{4} = \frac{5}{4} - (\frac{2}{4} - \frac{1}{4})$? (no)

f) Does $\frac{6}{12} - (\frac{4}{12} - \frac{3}{12}) = (\frac{6}{12} - \frac{4}{12}) - \frac{3}{12}$? (no)

g) Does (37 + 13) + 9 = 37 + (13 + 9) ? (yes)

h) Does 26 + (32 + 10) = (26 + 32) + 10 ? (yes)

i) Does (25 - 13) - 7 = 25 - (13 - 7) (no)

j) Does 75 - (50 - 25) = (75 - 50) - 25 ? (no)

k) Does $\frac{3}{4} + (\frac{1}{4} + \frac{2}{4}) = (\frac{3}{4} + \frac{1}{4}) + \frac{2}{4}$? (yes)

l) Does $(\frac{2}{6} + \frac{3}{6}) + \frac{1}{6} = \frac{2}{6} + (\frac{3}{6} + \frac{1}{6})$? (yes)

In the exercises above tell which examples illustrate the associative property. (a, b, g, h, k, l)

3\. Write the following expressions as decimal numerals.

Example a is done for you.

a) $7 + 1.6 + .05 = 8.65$

b) $2 + .3 + .06$ $(= 2.36)$

c) $5 + .2 + .17$ $(= 5.37)$

d) $21 + .4 + .22$ $(= 21.62)$

e) $9 + .8 + .23$ $(= 10.03)$

f) $16 + 1.6 + .16$ $(= 17.76)$

g) $3 + .2 + .75$ $(= 3.95)$

h) $61 + .3 + .81$ $(= 62.11)$

i) $8 + 2.5 + .52$ $(= 11.02)$

j) $19 + 9.7 + .36$ $(= 29.06)$

4\. Express each answer in its simplest form.

a) $\frac{3+1}{2+2} + \frac{4+1}{3+5} = n$ $(n = 1\frac{5}{8})$

b) $\frac{2+3}{1+2} - \frac{2+2}{3+3} = n$ $(n = 1)$

c) $\frac{4+1}{2+2} - \frac{2+3}{3+3} = n$ $(n = \frac{5}{12})$

d) $\frac{4+0}{1+2} + \frac{5+2}{5+7} = n$ $(n = 1\frac{11}{12})$

e) $\frac{4+1}{3+3} - \frac{2+1}{4+4} = n$ $(n = \frac{11}{24})$

f) $\frac{3}{4} + \frac{3}{8} = n$ $(n = 1\frac{1}{8})$

g) $\frac{3 \times 3}{2 \times 3} - \frac{1 \times 2}{1 \times 3} = n$ $(n = \frac{5}{6})$

h) $\frac{4+3}{4+2} + \frac{2+1}{3+1} = n$ $(n = 1\frac{11}{12})$

i) $n = \frac{2 \times 1}{3 \times 1} - \frac{3 \times 1}{5 \times 2}$ $(n = \frac{11}{30})$

j) $\frac{4}{5} + \frac{5}{8} = n$ $(n = 1\frac{17}{40})$

5\. Write as base ten numerals.

a) 24_{five} (14)

b) 312_{four} (54)

c) 64_{seven} (46)

d) 301_{five} (76)

e) 212_{three} (23)

f) 43_{eight} (35)

g) 43_{six} (27)

h) 322_{four} (58)

i) 441_{five} (121)

j) 645_{seven} (327)

6. Complete the chart below. Example a is worked for you.

Number Pair	All Factors	G C F	Multiples to L C M	L C M
a) 9	1, 3, 9	3	9, 18, 27, 36	36
12	1, 2, 3, 4, 6, 12		12, 24, 36	
b) 12	(1, 2, 3, 4, 6, 12)	(6)	(12, 24, 36)	(36)
18	(1, 2, 3, 6, 9, 18)		(18, 36)	
c) 15	(1, 3, 5, 15)	(5)	(15, 30, 45, 60, 75, 90, 105, 120)	(120)
40	(1, 2, 4, 5, 8, 10, 20, 40		(40, 80, 120)	
d) 24	(1, 2, 3, 4, 6, 8, 12, 24)	(6)	(24, 48, 72, 96, 120)	(120)
30	(1, 2, 3, 5, 6, 10, 15, 30		(30, 60, 90, 120)	
e) 18	(1, 2, 3, 6, 9, 18)	(9)	(18, 36, 54, 72, 90)	(90)
45	(1, 3, 5, 9, 15, 45)		(45, 90)	
f) 16	(1, 2, 4, 8, 16)	(4)	(16, 32, 48, 64, 80, 96, 112)	(112)
28	(1, 2, 4, 7, 14, 28)		(28, 56, 84, 112)	

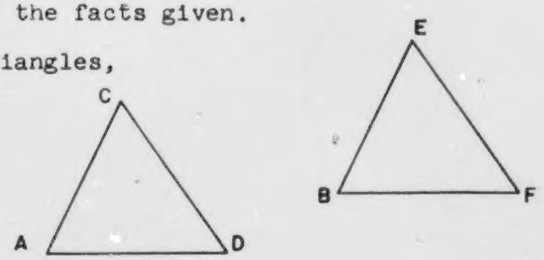
7. For your answers do not depend upon the appearance of the triangle. Use only the facts given.

a) In these two triangles, we know that

$\angle A \cong \angle B$

$\overline{AC} \cong \overline{BE}$

$\overline{AD} \cong \overline{BF}$



What do we know about $\angle D$ and $\angle F$, $\angle C$ and $\angle E$, also $\overline{CD}$ and $\overline{EF}$? $(\angle D \cong \angle F,\ \angle C \cong \angle E,\ \overline{CD} \cong \overline{EF})$

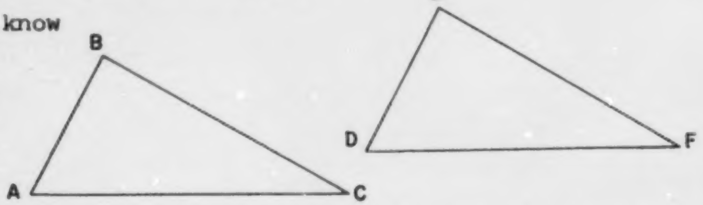
b) ABC is a scalene triangle.

We also know

$\overline{AB} \cong \overline{DE}$

$\overline{BC} \cong \overline{EF}$

$\overline{AC} \cong \overline{DF}$



What kind of a triangle would Δ DEF be? (Scalene)

List the pairs of congruent angles. ($\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$)

Part B

Write a mathematical sentence (or two sentences if necessary) for each problem and solve. Write an answer sentence.

1. A hostess made $1\frac{6}{8}$ gals. of punch for a party. She had $\frac{7}{8}$ gal. left. How much punch did the guests drink?
($1\frac{6}{8} - \frac{7}{8} = n$ or $\frac{7}{8} + n = 1\frac{6}{8}$, $n = \frac{7}{8}$ The guests drank $\frac{7}{8}$ gal.. punch.)
2. Wendy lives 8.6 mi. from her friend's house. One day she rode her horse part of the way to her friend's house. She walked the rest of the way. She walked 1.23 mi. How far did she ride before she started to walk to her friend's house? ($8.6 - 1.23 = n$, or $n + 1.23 = 8.6$, $n = 7.37$
Wendy rode 7.37 mi..)
3. In 1864 Abraham Lincoln was elected President for a second term. He received 2,216,067 votes. George McClellan ran against him and received 1,808,725 votes. How many fewer votes than Lincoln did McClellan receive?
($2,216,067 - 1,808,725 = n$; $n = 407,342$
McClellan received 407,342 fewer votes than Lincoln.)
4. When Sandra's father weighed her he said, "You weigh exactly 58 lbs. You have gained 1.8 lbs. since your birthday." How much did Sandra weigh on her birthday?
($58 - 1.8 = n$, $n = 56.2$ or $1.8 + n = 58$
Sandra weighed 56.2 lbs on her birthday.)

5. The Empire State Building was sold in 1951 for $51,000,000. This is three times the amount paid for the land on which it stands. How much did the land cost? ($51,000,000 \div 3 = n$ or $3 \times n = 51,000,000$; $n = 17,000,000$ The land cost $17,000,000.00..)

6. One week Bill worked for $\frac{3}{4}$ hr. on Monday, $1\frac{1}{2}$ hr. on Tuesday, $\frac{1}{2}$ hr. on Wednesday, $1\frac{2}{8}$ hr. on Thursday, and 2 hrs. on Friday. He is paid 65 cents an hour. How much did he make this week? [$(3/4 + 1\frac{1}{2} + \frac{1}{2} + 1\frac{2}{8} + 2) \times 65 = n$ or $3/4 + 1\frac{1}{2} + \frac{1}{2} + 1\frac{2}{8} + 2 = t$, $t = 6$, $6 \times 65 = n$; $n = 390$ Bill made $3.90..]

Individual Projects

1. You have developed rules for divisibility by the numbers 2, 3 and 5 and have given examples in which they are tested as factors of a number. Here are some rules for divisibility for you to test. You should try at least five examples to see if the rule is true.
   a) A number is divisible by 4 if two times the tens digit plus the units digit is divisible by 4.
   b) A number is divisible by 6 if the number is even and is divisible by 3.
   c) A number is divisible by 7 if the difference between twice the units digit and the number formed by omitting the units digit is divisible by 7.
   d) A number is divisible by 8 if four times the hundreds digit plus two times the tens digit plus the units digit is divisible by 8.
   e) A number is divisible by 9 if the sum of the digits is divisible by 9.

Review

SET II

Part A

1. Write the numerator and denominator as the product of primes. Example a is done for you.

a) $\frac{4}{16} = \frac{2 \times 2}{2 \times 2 \times 2 \times 2}$

b) $\frac{12}{42}$ $\left(\frac{2 \times 2 \times 3}{2 \times 3 \times 7}\right)$

c) $\frac{63}{81}$ $\left(\frac{3 \times 3 \times 7}{3 \times 3 \times 3 \times 3}\right)$

d) $\frac{15}{24}$ $\left(\frac{3 \times 5}{2 \times 2 \times 2 \times 3}\right)$

e) $\frac{10}{25}$ $\left(\frac{2 \times 5}{5 \times 5}\right)$

f) $\frac{12}{8}$ $\left(\frac{2 \times 2 \times 3}{2 \times 2 \times 2}\right)$

2. Select the symbols that represent zero from each row.

a) [✓ $\frac{3}{5} - \frac{6}{10}$], $421 + 0$, [✓ $0 \div 6$], [✓ $\frac{6 - 6}{3}$], [✓ 12×0].

b) $\frac{8}{8}$, [✓ $\frac{4 - 4}{2 - 2}$], [✓ 0×364], $\frac{12}{8} - \frac{3}{4}$, $75 - 0$

c) [✓ 0×6], [✓ $5 - \frac{10}{2}$], [✓ $(2 \times 3) \times 0$], $\frac{3}{3}$, [✓ $\frac{8 - 8}{6}$]

d) [✓ $10.6 - 10.60$], $0 + 982$, [✓ 7×0], $\frac{6 + 2}{8}$, [✓ $0 \div 12$]

3. Copy and place parentheses to make each a true statement, then solve. Example a is shown.

a) $(3 \times 5) + 7 = 20 + 2,\ 22 = 22$

b) $18 \times (3 + 3) = 108$ $(108 = 108)$

c) $(32 \div 8) + 4 = 6 + 2$ $(8 = 8)$

d) $44 = 50 - (12 \div 2)$ $(44 = 44)$

e) $72 \div (4 \times 2) = (6 - 2) + 5$ $(9 = 9)$

f) $48 + (10 \div 2) = (6 \times 10) - 7$ $(53 = 53)$

g) $7 + (49 \div 7) = (12 - 2) + 4$ $(14 = 14)$

h) $15 \times (12 \div 6) = (24 \div 4) \times 5$ $(30 = 30)$

i) $45 \times (24 \div 6) = (54 \times 3) + 18$ $(180 = 180)$

j) $(31 \times 22) + 18 = (35 \times 40) \div 2$ $(700 = 700)$

4. Rename the following in simplest mixed form.

a) $\frac{7}{2}$ $(3\frac{1}{2})$
b) $\frac{14}{3}$ $(4\frac{2}{3})$
c) $\frac{17}{4}$ $(4\frac{1}{4})$
d) $\frac{15}{6}$ $(2\frac{1}{2})$
e) $\frac{14}{4}$ $(3\frac{1}{2})$
f) $\frac{21}{9}$ $(2\frac{1}{3})$
g) $\frac{17}{5}$ $(3\frac{2}{5})$
h) $\frac{18}{4}$ $4\frac{1}{2}$

5. Copy and replace n with the number n represents. An example is shown.

a) $\frac{3}{4} = \frac{n}{16}$, $\frac{3}{4} = \frac{12}{16}$
b) $\frac{2}{3} = \frac{n}{15}$, $(\frac{2}{3} = \frac{10}{15})$
c) $\frac{7}{5}$ $\frac{21}{n}$, $(\frac{7}{5} = \frac{21}{15})$
d) $\frac{n}{4} = \frac{20}{16}$, $(\frac{5}{4} = \frac{20}{16})$
e) $\frac{5}{n} = \frac{15}{24}$, $(\frac{5}{8} = \frac{15}{24})$
f) $\frac{n}{3}$ $\frac{12}{9}$, $(\frac{4}{3} = \frac{12}{9})$
g) $\frac{3}{6} = \frac{n}{18}$, $(\frac{3}{6} = \frac{9}{18})$
h) $\frac{8}{7} = \frac{32}{n}$, $(\frac{8}{7} = \frac{32}{28})$

6. Use the largest multiple of 10, 100, or 1,000 to make each of these true sentences. Example a is shown.

a) $9 \times \underline{7000} < 63{,}801$
b) $20 \times \underline{90} < 1{,}848$
c) $4{,}328 > \underline{600} \times 7$
d) $\underline{300} \times 30 < 10{,}380$
e) $5{,}161 > 6 \times \underline{800}$
f) $53{,}871 > \underline{6000} \times 8$
g) $80 \times \underline{9000} < 764{,}892$
h) $\underline{70} \times 7 < 535$
i) $38{,}462 > 90 \times \underline{400}$
j) $\underline{3000} \times 30 < 96{,}483$

7. Which of these numbers are equal to $1\frac{1}{2}$?

1.5 , $\frac{7}{5}$, 1.05 , $\frac{9}{6}$, $\frac{12}{8}$ (1.5 , $\frac{6}{8}$, $\frac{12}{8}$)

Which of these numbers are equal to 3?

$\frac{12}{3}$, $2 + \frac{5}{5}$, $1\frac{8}{4}$, $\frac{24}{6}$, 2.10 ($2+\frac{5}{5}$, $1\frac{8}{4}$)

Which of these numbers are equal to $\frac{3}{4}$?

$\frac{9}{12}$, $8 - 7\frac{1}{2}$, $.75$, $\frac{15}{20}$, $\frac{8}{12}$ ($\frac{9}{12}$, $.75$, $\frac{15}{20}$)

Which of these numbers are greater than $\frac{2}{3}$?

$\frac{9}{12}$, $\frac{9}{15}$, $.7$, $.285$, $\frac{5}{9}$ ($\frac{9}{12}$, $.7$)

Which of these numbers are less than $\frac{5}{6}$?

$.7$, $\frac{11}{12}$, $.90$, $\frac{3}{4}$, $.065$ ($.7$, $\frac{3}{4}$, $.065$)

8. Use the word plane, line, line segment, ray, circle or quadrilateral to complete these statements.

a) A sheet of paper could be thought of as a model of a (plane).

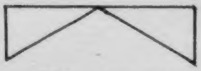
b) A clothes line stretched tightly between two poles could be thought of as a model of a (line segment).

c) A wedding ring could be thought of as a model of a (circle).

d) A window frame could be thought of as a model of a (Quadrilateral).

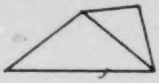
e) The beam of light from a spotlight could be thought of as a model of a (ray).

9. Construct two congruent triangles with sides whose lengths in inches are 3, 2, and 4 and whose intersection is one vertex. The interior of one triangle should be in the exterior of the other.



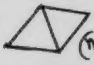
(not to scale, also, answers may vary)

10. Construct two triangles with one common side. One triangle has sides whose length in inches are 5, 3 and 3. The second triangle has sides whose lengths in inches are 2, 2 and 3. The interior of one will be in the interior of the other.



(not to scale, answers will vary)

11. Construct two triangles with one common side. One triangle has sides whose lengths in inches are 4, 4 and 6. The second triangle has sides whose lengths in inches are 3, 3 and 4. The interior of the second triangle will be in the exterior of the first.



(not to scale, answers will vary)

Part B

Write a mathematical sentence (or two sentences if necessary) for each problem and solve. Write an answer sentence.

1. Mary has $\frac{1}{2}$ c. flour. She looked at three recipes for cupcakes, one uses $\frac{7}{24}$ c. flour, another uses $\frac{2}{6}$ c. flour, and the third uses $\frac{4}{8}$ c. flour. Which recipe will use all of her flour? ($\frac{1}{2} = \frac{12}{24}$; $\frac{7}{24}$; $\frac{2}{6} = \frac{8}{24}$, $\frac{4}{8} = \frac{12}{24}$, $\frac{4}{8} = \frac{1}{2} = \frac{12}{24}$
The third recipe uses $\frac{4}{8}$c. or $\frac{1}{2}$c flour, all of her flour.)

2. The German bobsledding team made a trial run in 5 minutes, 07.84 seconds. The United States bobsledding team made their trial run in 5 minutes, 20.1 seconds. Which team made the faster time? How much faster? (20.1 − 07.84 = n; n = 12.26 The German team made better time, 12.26 seconds faster.)

3. A parking lot has 24 rows for cars. Each row holds 32 cars. How many cars are in the lot when the rows are filled?
($24 \times 32 = n$, $n = 768$ There are 768 cars in the lot.)

4. In the Soap Box Derby John finished his run in 2.7 min. Mac finished in 2.68 min., and Terry finished in 2.07 min. What was the difference in time between the fastest and slowest runs? ($2.70 - 2.07 = n$, $n = .63$
Terry's time was .63 min. faster than John's.)

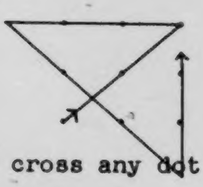
5. If the speed of a meteroid moving through space averages 30 miles per second, what will be its average speed per hour? ($30 \times 60 \times 60 = n$ or $30 \times 60 = t$, $t \times 60 = n$, $n = 108{,}000$
The meteoroid will have an average speed of 108,000 m.p.h..)

6. A gallon of water weighs 8.33 lbs. It is carried in a bucket weighing 1.8 lbs. What is the total weight of the gallon of water and the bucket? ($1.8 + 8.33 = n$ $n = 10.13$
The total weight will be 10.13 lbs..)

Puzzles

1. What number base is used in each of these?
   a) Jan said, "My cat weighs 2 pounds, or $112_{\underline{five}}$ ounces." (Base Five)
   b) My little sister is $100_{\underline{two}}$ years old. In one year she will be entering the first grade. (Base Two)
   c) The teacher is five feet six inches or $73_{\underline{nine}}$ inches tall. (Base Nine)

2. Cross out every dot with four line segments. Do not lift the pencil from the paper until all nine dots are crossed. Do not retrace a line or cross any dot more than once.



Review

SET III

Part A

1. Arrange the following numbers in order from least to greatest.
   a) $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{4}$, $\frac{1}{5}$ ($\frac{1}{6}$, $\frac{1}{5}$, $\frac{1}{4}$, $\frac{1}{3}$. $\frac{1}{2}$)
   b) $\frac{3}{8}$, $\frac{7}{8}$, $\frac{4}{8}$, $\frac{1}{8}$, $\frac{5}{8}$ ($\frac{1}{8}$. $\frac{3}{8}$, $\frac{4}{8}$, $\frac{5}{8}$. $\frac{7}{8}$)
   c) .7 , 1.4 , .73 , .29 , .4 (.29, .4, .7, .73, 1.4)
   d) $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{8}$, $\frac{1}{6}$, $\frac{1}{4}$ ($\frac{1}{6}$, $\frac{1}{4}$, $\frac{3}{8}$, $\frac{1}{2}$, $\frac{2}{3}$)
   e) $\frac{5}{12}$, $\frac{3}{8}$, $\frac{11}{24}$, $\frac{7}{12}$, $\frac{5}{6}$ ($\frac{3}{8}$. $\frac{5}{12}$, $\frac{11}{24}$, $\frac{7}{12}$, $\frac{5}{6}$)
   f) .29 , .029 , 2.9 , 29 , .5 (.029, .29, .5, 2.9, 29)
   g) .8 , .46 , 2, .059 , .4 (.059, .4, .46, .8 , 2)
   h) 51 , .5 , 5.01 , 5.1 , .51 (.5, .51 , 5.01, 5.1 , 51)

2. Using the symbol > , = , or < make the following true sentences.
   a) 28 x 2 < 154 + 88
   b) 24 x (24 x 6) = 6 x (24 x 24)
   c) 32 x 127 < 4,060 + 24
   d) $\frac{4}{10}$ = .40
   e) $\frac{3}{5}$ > $\frac{6}{12}$
   f) 31,106 > 74 x 419
   g) 47 x 608 = 28,576
   h) $\frac{2}{5} + \frac{1}{2} + \frac{2}{5}$ < 1.4
   i) .6 + 2.15 + .25 > 2.9
   j) 2,605 ÷ 56 = 46 r 29

3. Write fraction names for these numbers. Example a is done for you.

a) $5\frac{1}{3} = \frac{16}{3}$
b) 1.1 $\left(\frac{11}{10}\right)$
c) $8\frac{3}{5}$ $\left(\frac{43}{5}\right)$
d) $7\frac{7}{9}$ $\left(\frac{70}{9}\right)$
e) 2.53 $\left(\frac{253}{100}\right)$
f. .73 $\left(\frac{73}{100}\right)$
g) $7\frac{1}{4}$ $\left(\frac{29}{4}\right)$
h) $25\frac{1}{2}$ $\left(\frac{51}{2}\right)$
i) 4.8 $\left(\frac{48}{10}\right)$
j) $12\frac{4}{8}$ $\left(\frac{100}{8}\right)$

4. Without working the problem, tell which expression in each row represents the largest number.

a) [✓ 253 ÷ 15,] 253 ÷ 26, 253 ÷ 19, 253 ÷ 39
b) 341 × 23, 314 × 23, 336 × 23, [✓ 364 × 23]
c) $\frac{4}{8} + \frac{1}{3}$, $\frac{4}{8} + \frac{3}{3}$, [✓ $\frac{4}{8} + \frac{4}{3}$,] $\frac{4}{8} + \frac{2}{3}$
d) 1,192 ÷ 28, 1,301 ÷ 28, 1,099 ÷ 28, [✓ 1,900 ÷ 28]
e) $\frac{2}{3} - \frac{1}{2}$, [✓ $\frac{2}{3} - \frac{1}{4}$,] $\frac{2}{3} - \frac{4}{6}$, $\frac{2}{3} - \frac{5}{8}$

5. In the above examples find the expression in each row that names the smallest number. (a) 253 ÷ 39 ; b) 314 × 23 ; c) $\frac{4}{8} + \frac{1}{3}$ d) 1099 ÷ 28 e) $\frac{2}{3} - \frac{4}{6}$)

6. Write the greatest common factor for each pair.

a) 28 , 35 (7).
b) 40 , 54 (2)
c) 27 , 54 (27)
d) 18 , 60 (6)
e) 25 , 120 (5)
f) 72 , 30 (6)
g) 12 , 84 (12)
h) 42 , 70 (14)
i) 225 , 45 (45)
j) 33 , 363 (33)

7. Copy and place parentheses to make each a true statement.

a) $18 \times 23 \div 9 = 9 \times 92 \div 18$ $[(18 \times 23) \div 9 \quad (9 \times 92) \div 18]$

b) $31 \times 50 \div 2 = 24 \times 33 - 17$ $[31 \times (50 \div 2) = (24 \times 33) - 17]$

c) $64 \times 23 - 4 > 26 + 17 \times 43$ $[(64 \times 23) - 4 > 26 + (17 \times 43)]$

d) $42 \times 24 + 3 \neq 21 \times 54$ $[(42 \times 24) + 3 \neq 21 \times 54]$

e) $43 - 9 \times 16 = 4 + 27 \times 20$ $[(43 - 9) \times 16 = 4 + (27 \times 20)]$

f) $36 + 18 \times 47 < 48 \times 12 + 8$ $[36 + (18 \times 47) < 48 \times (12 + 8)]$

g) $27 \times 96 + 8 = 13 \times 196 + 4$ $[(27 \times 96) + 8 = 13 \times (196 + 4)]$

h) $65 \times 64 + 30 > 36 \times 113 + 8$ $[(65 \times 64) + 30 > (36 \times 113) + 8]$

8. Use closed or not closed to complete and make these true sentences.

a) The set of whole numbers is (closed) under addition.

b) The set of odd numbers is (not closed) under subtraction.

c) The set of counting numbers is (closed) under addition.

d) The set of whole numbers is (closed) under multiplication.

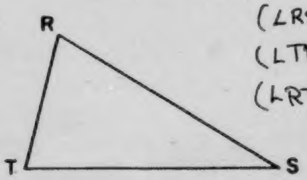
e) The set of whole numbers is (not closed) under division.

f) The set of counting numbers from 23 to 75 is (not closed) under addition.

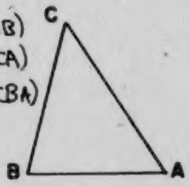
g) The set of whole numbers is (not closed) under subtraction.

h) The set of counting numbers less than 43 is (not closed) under multiplication.

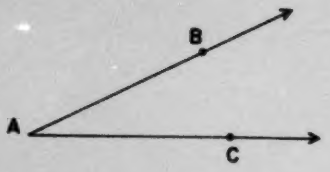
9. Copy and compare the sizes of ∠RST and ∠CAB, ∠TRS and ∠BCA, ∠RTS and ∠CBA.

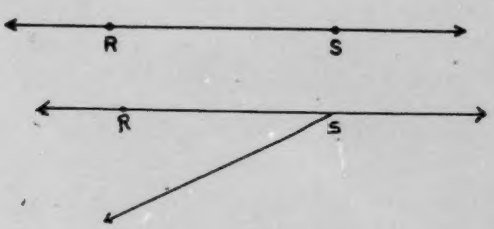


(∠RST is smaller than ∠CAB)
(∠TRS is larger than ∠BCA)
(∠RTS is congruent to ∠CBA)



10. Use your compass and straightedge to copy $\angle BAC$ on $\overleftrightarrow{RS}$ so that point S corresponds to A and $\overrightarrow{AB}$ falls on $\overrightarrow{SR}$.





Part B

Write a mathematical sentence (or two sentences if necessary) for each problem and solve. Write an answer sentence.

1. There are 40 pages in each Orange Trading Stamp Book. Each page holds 35 stamps. How many stamps will be needed to fill one book? ($35 \times 40 = n$, $n = 1400$
1400 stamps will be needed to fill one book..)

2. Smith's Department Store ordered 1608 Christmas tree ornaments from Japan. They arrived in 67 boxes with the same number of ornaments in each box. How many ornaments were in each box? ($1608 \div 67 = n$, $n = 24$
There are 24 ornaments in each box.)

3. The food committee for the class picnic ordered hamburgers. Five-eighths of the class wanted hamburgers with onions. What part of the class wanted theirs without onions?
($1 - \frac{5}{8} = n$, $n = \frac{3}{8}$ $\frac{3}{8}$ of the class wanted theirs without onions.)

4. On its picnic, the class took $2\frac{1}{3}$ gals. ice cream. They used $1\frac{2}{5}$ gals. for sundaes. How much ice cream was left for cones? ($2\frac{1}{3} - 1\frac{2}{5} = n$, $n = \frac{14}{15}$
They had $\frac{14}{15}$ gal. ice cream left for cones.)

5. The Campfire Girls in one town sold 426 boxes of candy the first week of their sale, 281 boxes the second week, and 469 boxes the third week. What was the average number of boxes sold each day? [$(426 + 281 + 469) \div 21 = n$ or $426 + 281 + 469 = p$
$p \div 21 = n$; $n = 56$]
(The girls sold an average of 56 boxes a day.)

6. Three frying chickens weigh $1\frac{3}{4}$ pounds, $2\frac{1}{2}$ pounds and $2\frac{1}{4}$ pounds. What is their total weight? ($1\frac{3}{4} + 2\frac{1}{2} + 2\frac{1}{4} = w$, $w = 6\frac{1}{2}$
Their total weight is $6\frac{1}{2}$ lbs..)

7. The speedometer of a car shows 74,286.1 miles at the end of the month. The car had gone 3,729.4 miles that month. What had the speedometer shown at the beginning of the month?
($74,286.1 - 3,729.4 = r$, $r = 70,556.7$
The speedometer had shown 70,556.7 miles.)

8. A box factor makes 2,940 soap boxes in one hour. How many dozen boxes are made in one continuous eight hour shift?
[$(2940 \times 8) \div 12 = t$ or $(2940 \div 12) \times 8 = t$ or $2940 \times 8 = d$, $d \div 12 = t$, $t = 1960$]
(There are 1960 dozen boxes made in eight hours.)

Individual Projects

1. Make a model of a geometric prism. The measure of the shortest edge should be no less than 3 inches. Color the faces so that none of the faces with a common edge are the same color. Display it for your class.

2. Make a model of a polygon. Use wire for the line segments. The measure of the shortest segment should be no less than 4 inches. You could use two or more of these to make an interesting mobile for your class.

3. Many great men have made important contributions to mathematics. Make a report about one of these famous mathematicians and his contributions.

Chapter 7

MEASUREMENT OF ANGLES

PURPOSE OF UNIT

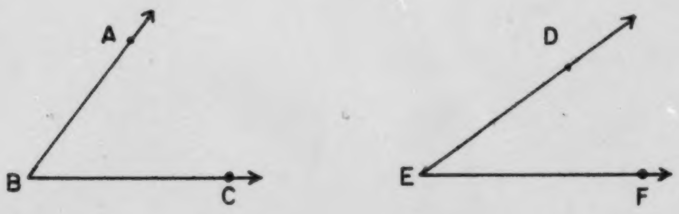
The purpose of this unit is to develop understanding and skill in the measurement of angles. Like the study of linear measurement, the study of angular measurement may be divided into four major considerations which parallel the historical development:

1. Intuitive awareness of difference in size.
2. Choice of an arbitrary unit with the understanding that the unit must be of the same nature as the thing to be measured--a line segment to measure a line segment, an angle to measure an angle, etc.
3. Selection of a standard unit for purposes of communication.
4. Designing a suitable scale for convenience in measuring.

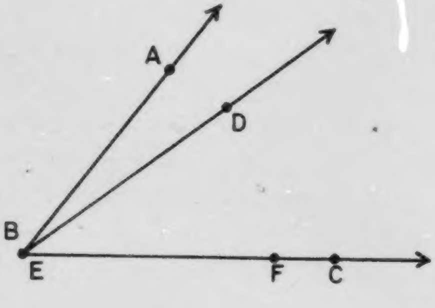
MATHEMATICAL BACKGROUND

It is of interest to note that in the development of the idea of measurement for the pupils, we are actually following the historical development of this concept. The counting of discrete or separate objects (like finding the number of sheep in a herd) was not a technique applicable to the measure of a continuous curve (like determining the length of the boundaries of a wheat field). At first the notion of size was realized intuitively. One boundary was longer than another; one piece of land was larger than another. This sufficed until fields bordered more closely on each other and more refined measures were necessary. Then a unit of measure (e.g. that part of a rope between two knots) was agreed upon. Now it was possible to designate a piece of property as having a length of "50 units of rope" and having a width of "30 units of rope." With the increase of travel and communication, it became obvious that "50 units of rope" did not represent the same length to all people unless they were familiar with the unit. The need for a standard unit arose. Once a standard unit was agreed upon, a scale was devised for greater convenience in measuring.

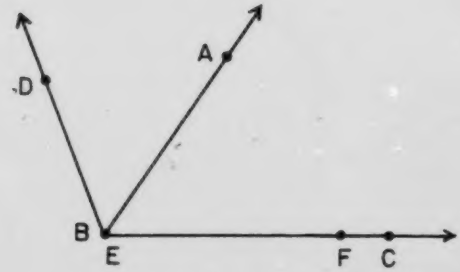
Basing the development of the concept of angle measurement upon these four considerations, appeal is first made to the pupil's intuition in making comparisons of the sizes of angles. Recall from Ch. 4 (Congruence of Common Geometric Figures) that an angle is a set of points consisting of two rays with a common endpoint, but not both on the same line. Consider any two angles, say $\angle ABC$ and $\angle DEF$, as pictured below.



We can conceive of placing these angles one on top of the other in such a way that $\overrightarrow{BC}$ and $\overrightarrow{EF}$ coincide while $\overrightarrow{BA}$ and $\overrightarrow{ED}$ both extend above $\overrightarrow{BC}$, thus.



In the example illustrated above, $\overrightarrow{ED}$ extends into the interior of $\angle ABC$. We say that $\angle DEF$ is smaller than $\angle ABC$. If instead it happens that $\overrightarrow{ED}$ coincides with $\overrightarrow{BA}$, we say that $\angle DEF$ is of the same size as $\angle ABC$. (Recall that in this case we also say that $\angle AEC$ and $\angle DEF$ are congruent.) If $\overrightarrow{ED}$ extends into the exterior of $\angle ABC$, as shown below, we say $\angle DEF$ is greater than $\angle ABC$.



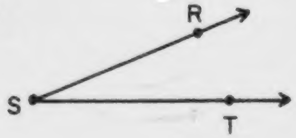
Thus, given ∠ ABC and ∠ DEF, exactly one of the following three statements is true:

1. ∠ DEF is of smaller size than ∠ ABC.
2. ∠ DEF is of the same size as ∠ ABC.
3. ∠ DEF is of greater size than ∠ ABC.

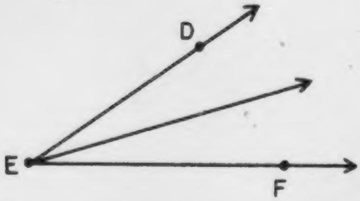
Just as we think of every line segment as having a certain exact length, so too we think of every angle as having a certain exact size, even though this size can be determined only approximately by measuring a chalk or pencil drawing representing it.

Let us examine this process of angular measurement more closely. As in the case of linear measurement, the first step is to choose a certain angle to serve as unit. This means that we select an angle and agree to consider its size to be described or measured, exactly, by the number 1. Call this ∠ RST.

Unit Angle

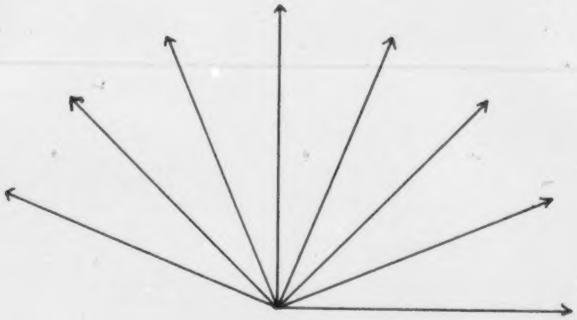


Now we can conceive of forming an angle DEF by laying off the unit ∠ RST twice about a common vertex E as suggested in the picture below.

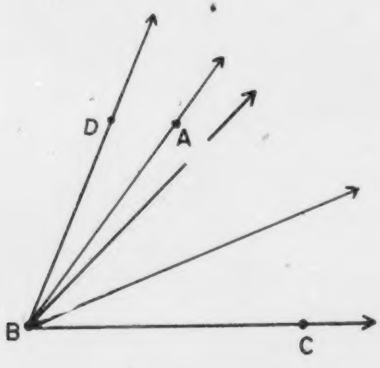


We say that $\angle DEF$ has size exactly 2 units, although $\angle DEF$ can be represented only approximately by a drawing.

In similar fashion we can conceive of forming an angle of size exactly 3 units, or exactly 4 units, and so forth, until we have drawn an angle whose interior is nearly half a plane, as shown below.



We also can conceive of an angle, call it $\angle ABC$, such that the unit $\angle RST$ will not fit into $\angle ABC$ a whole number of times. In the picture below we have shown an $\angle ABC$ such that, starting at $\overrightarrow{BC}$ the unit $\angle RST$ can be laid off 2 times about B without quite reaching $\overrightarrow{BA}$, though if we were to lay off the unit 3 times we would arrive at a ray, call it $\overrightarrow{BD}$, which is well beyond $\overrightarrow{BA}$.



What can we say about the size of $\angle ABC$? Well, we can surely say that $\angle ABC$ has size greater than 2 units and less than 3 units. In the particular case pictured we can also estimate by eye that the size of $\angle ABC$ is nearer to 2 units than to 3 units, so we can say that *to the nearest unit* $\angle ABC$ has size 2 units. This is the best we can do without considering fractional parts of units, or else shifting to a smaller unit.

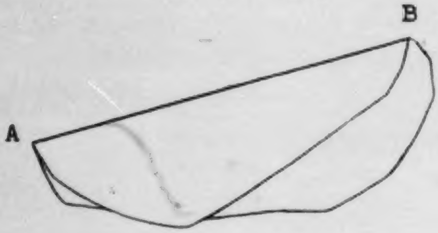
When a unit is agreed upon, then a scale may be devised to facilitate measurement. If we decide to use a segment of one inch as a unit of *linear* measure, then a straightedge may have successive congruent segments of unit length laid off on it (each segment intersecting adjacent segments in one endpoint only). If we further associate each endpoint from the first to the last on the straightedge, with the whole numbers taken in order, (0, 1, 2, 3, 4, 5, 6, ...) then we have established a scale for linear measure.

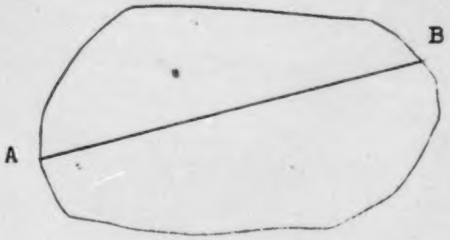
We follow the same procedure in setting up a device for angle measurement - a protractor. For the pupil's first introduction to the use of a protractor, the unit chosen is a large one. Thus, attention, properly directed to the correct use of the protractor, is not diverted by problems due to difficulty in reading a closely marked scale. Since this first selection of a unit angle to be laid off successively on the half plane is not a standard unit, we may name it whatever, we choose, even an "octon".

The choice of the octon as a unit angle is arbitrary but not accidental. It was selected so that if eight congruent angles, each 1 octon in size, are laid out successively with a common vertex then they together with their interiors will exactly cover a half-plane. The following demonstration shows how to use a paper folding to determine the size of the octon.

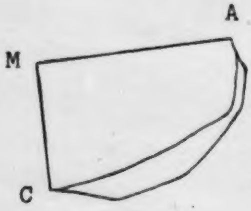
Select any piece of paper (it might even be irregularly shaped).

Fold it once to make a model of a line separating two half-planes. Call it $\overleftrightarrow{AB}$.

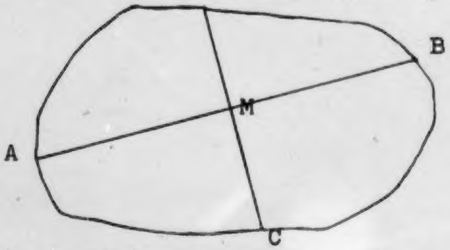




Choose a point M on $\overleftrightarrow{AB}$ and fold through M so that $\overleftrightarrow{MA}$ falls on $\overleftrightarrow{MB}$.

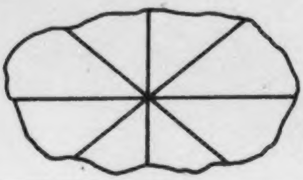


The $\angle$ AMC is a model of a right angle.
If you unfold the paper, it will appear like this.



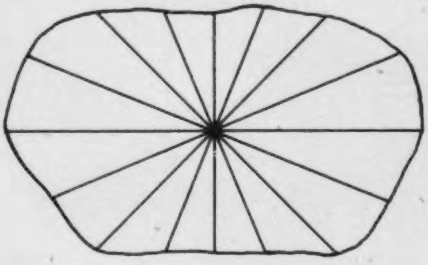
This shows four models of angles, all congruent, which together with their interiors, fill the plane.

Refold the paper so that you again have a model of a single right angle. Now fold so that the rays represented by $\overline{AM}$ and $\overline{CM}$ coincide.



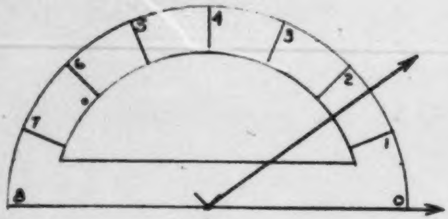
This provides us with a model of an angle such that any four successive angles with a common vertex will exactly fit in the half-plane.

Refold your paper. Proceed to make one more fold as before. You now have a model of an angle of one octon, eight of which, successively placed with a common vertex, will exactly fit on the half-plane and its edge.

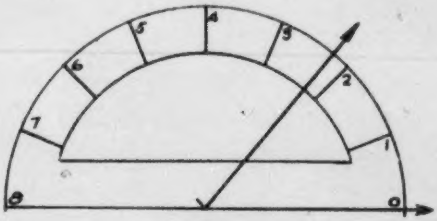


Each ray of the successively marked-off octons is associated with a whole number, taken in order from 0 to 8. We now have a protractor with a scale on it suitable for use in measuring angles. It should be emphasized that the measure of an angle is a number. We read $m\angle ABC = 7$ as "The measure of angle ABC is seven." If the unit is the octon, then we understand the statement to mean: "The measure of $\angle ABC$, in octons, is seven." We cannot say, "$\angle ABC = 7$" because $\angle ABC$ is a set of points and 7 is a numeral. But the measure of $\angle ABC$ is a number, so that a statement like: "$m\angle ABC = 7$" is permissible, since we have numerals on both sides of the " = " symbol.

Eventually the pupil recognizes that approximate readings of angle measure "to the nearest octon" leads him into a situation in which both ∠ A and ∠ B (clearly not the same size) have a measure of 2 to the nearest octon.

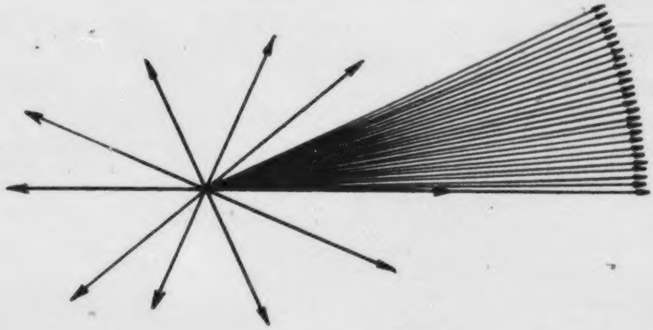


A



B

The need for a smaller unit is appreciated. For purpose of communication, a standard unit has been agreed upon. We call the size of the standard unit of angle measure, one degree, and write it in symbols as 1°. When we speak of the size of an angle we may say its size is 45°, but if we wish to indicate its measure we must keep in mind that a measure is a number and say that its measure, in degrees, is 45. If we lay off 360 of these unit angles using a single point as a common vertex, then these angles together with their interiors cover the entire plane.



Even in ancient Mesopotamian civilization the angle of 1° as the angle of unit measure, was used. The selection of a unit angle which could be fitted into the plane (as above) just 360 times was probably influenced by their calculation of the number of days in a year as 360.

In this unit we concern ourselves only with angles whose measures are between 0 and 180. Because of our definition of an angle and its interior it is not possible to have an angle whose rays coincide or extend in a straight line. Subsequent extension in later grades of the definition of angle will make it possible to discuss an angle of any size.

TEACHING THE UNIT

The lessons in this unit vary in their composition. Some have three parts which are: first, Suggested Teaching Procedure, second, Exploration, and third, Exercises which the children should do independently. In some lessons the Exploration and Exercises are sufficient to develop the lesson. Some lessons need only the Exploration to clarify the concepts for the children.

The first part Suggested Teaching Procedure provides an overview of the lesson. It is here that the teacher will find suggestions for providing the background the children will need for the understandings and skills to be developed.

Some teachers may prefer to have the children's books closed during this introduction of the concepts. During the second part of the lesson, the Exploration in the pupil's book, the pupils and teacher will read and answer the questions together. She may say, for example, "Now turn to page — and look at the Exploration. Is this what we did? Is this what we found to be true?" A resourceful teacher will be sensitive to the mood of her class and will not extend this part of the lesson beyond the point of interest.

Other teachers may go immediately to the Explorations. The Exploration then serves as a guide for the lesson. Still others may wish to have the pupil's book closed during the presentation and then have the pupils read the Exploration independently for review.

The third part of the lesson is the Independent Exercises. These are designed for the pupil to work independently. They are provided for maintenance and establishment of skill but they are also developmental in nature and help pupils gain additional understandings and skills.

Each teacher should feel free to adapt these ideas in a way that will suit her method of teaching and in a way that meets the particular needs of her class.

The first section of this unit is a review of material covered in the SMSG text for the fourth grade. If the pupils have not studied this material, you will need to spend more time on this section. In either case, you should have a copy of the SMSG text for grade four.

References:

1. School Mathematics Study Group Text for Grade Four.
2. *Mathematics for Junior High School*, Volume I, Chapter IV, School Mathematics Study Group.
3. Freeman, Mae and Ira, *Fun with Figures*, New York: Random House, 1946.
4. Ravielli, A., *An Adventure in Geometry*, New York: Viking Press.
5. Bassetti, F., Solid Shapes Lab, New York Science Material Center.
6. R. D. Anderson, *Concepts of Informal Geometry*, Volume V, Studies in Mathematics, School Mathematics Study Group.

UNIT SEGMENTS AND UNIT ANGLES

Objective: To develop the following understandings and skills.

1. The unit used for measuring an angle is an angle.
2. The unit angle is chosen arbitrarily.
3. The exact measure of a given angle in terms of a unit is the number (not necessarily a whole number) of times the unit angle will fit into the given angle.
4. The approximate measure of a given angle in terms of a unit is the nearest whole number of times the unit angle will fit into the given angle.

Materials:

Teacher: Chalkboard or string compass, straightedge (meter stick or yard stick), sheet of plastic for tracing, colored chalk.

Pupil: Compass, straightedge, tracing paper.

Vocabulary: measure, unit

If there is an interval between the study of EB-114 and this unit, review the definitions of line, ray, line segment, angle, and congruence. Work at the chalkboard while the pupils work on paper at their seats. Follow the Exploration as closely as possible. In Exercise 1 on Page 1, be sure the pupils choose a segment MN small enough so that it can be laid off at least three times on the rays as in Exercise 3. Stress that the measure is a number. Hence $m\ \overline{AB} = 3$ and not $m\ \overline{AB} = 3$ units. "3 units" is *not* a number but a description of a physical quantity. Similarly $m\angle R = 5$ and *not* $m\angle R = 5$ units.

Use colored chalk to help pupils in Exercises 7-10 of the Exercises in visualizing the overlapping angles whose interiors intersect.

Chapter 7

MEASUREMENT OF ANGLES

UNIT SEGMENTS AND UNIT ANGLES

Exploration

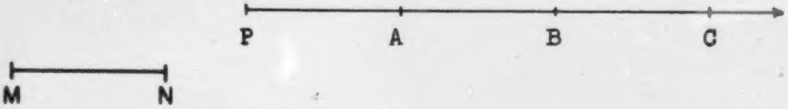
You have studied congruent angles, and you know that congruent angles have the same size. You have learned also how to tell which of two angles has the larger size. But we need to have a way to describe the size of an angle more exactly, that is, to *measure* an angle. Let us see how this could be done.

Recall how you found a method to measure a line segment. See if what you did to measure a segment suggests how an angle might be measured. Read the instructions of examples 1, 2, and 3 before you start the drawing requested in example 1.

1. Draw a ray on your paper. Call its endpoint P.
   Also draw a short segment not on the ray. Call it $\overline{MN}$.

2. On your ray construct a segment congruent to $\overline{MN}$, with one endpoint P. Call it $\overline{PA}$.

3. On the ray, construct a second segment congruent to $\overline{MN}$, with A as endpoint. Call it $\overline{AB}$. On the ray construct a third segment congruent to $\overline{MN}$. Call it $\overline{BC}$. Your drawing should look like this.



4. Copy and complete the following statements. Look at $\overline{MN}$ and $\overrightarrow{PC}$ you have drawn on your paper. Call the length of $\overline{MN}$ one unit. Then

a) the length of $\overline{PA}$ is (1) unit.

b) the length of $\overline{AB}$ is (1) unit.

c) the length of $\overline{BC}$ is (1) unit.

d) the length of $\overline{PB}$ is (2) units.

e) the length of $\overline{PC}$ is (3) units.

f) the length of $\overline{AC}$ is (2) units.

The number 2 is called the measure of $\overline{PB}$.

5. What is the measure of $\overline{PC}$? (3) of $\overline{AC}$? (2) of $\overline{AB}$? (1)

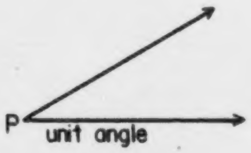
6. Did the pupil next to you make $\overline{MN}$ the same length?

7. If you are told only the measure of a segment can you know how long it is? (No) What else must you know? (You must know the unit segment used.)

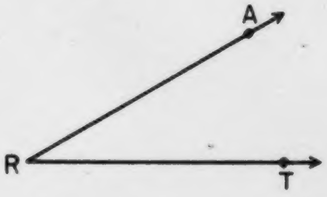
8. Choose a new segment, different from $\overline{MN}$, as your unit. Construct a segment whose measure, using this new unit, is 4. Construct another segment whose measure is 3.

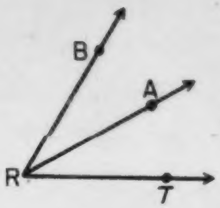
9. You used a line segment as a unit to measure line segments. What should you use as a unit to measure an angle? (An angle)

10. Use $\angle P$ as a unit angle. Draw $\overrightarrow{RT}$ on a sheet of paper. Make a tracing of $\angle P$ on thin paper. Place the tracing with P on R and one side of $\angle P$ on $\overrightarrow{RT}$. Then use the sharp end of your compass to mark a point A through the tracing to your drawing. Remove the tracing and draw $\overrightarrow{RA}$. Is $\angle ART \cong \angle P$? (Yes)



11. What is the measure of $\angle ART$? (The measure of $\angle ART$, using $\angle P$ as the unit angle, is 1.)





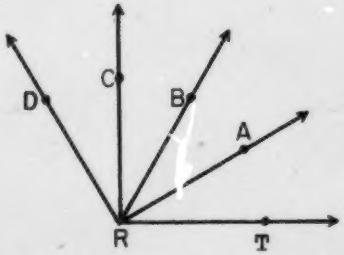
12. On your drawing of ∠ ART, place the tracing of the unit ∠ P so P is on R and one side of ∠ P is on $\overrightarrow{RA}$ and the other side of ∠ P is in the exterior of ∠ ART. Use the sharp end of your compass to mark a point B, and draw $\overrightarrow{RB}$. Is ∠ ARB ≅ ∠ P? (yes)

13. Using ∠ P as the unit angle, what is the measure of ∠ ARB? (1) What is the measure of ∠ BRT? (2)

14. Continue as in Problem 12.

a) Place the tracing on the unit ∠ P with P on (R) and one side of ∠ P on ($\overrightarrow{RB}$). Be sure to place the tracing so the other side of ∠ P is in the (exterior) of ∠ BRT. Use the sharp end of your compass to mark a point C, through your tracing to your drawing. Remove the tracing and draw $\overrightarrow{RC}$.

b) Repeat this process one more time in order to draw ∠ CRD. Your drawing should now look like this:



15. What is the measure in unit angles of ∠ CRT? (3) ∠ CRA? (2) ∠ CRB? (1) ∠ CRD? (1)

16. Since you and all your classmates used the same unit angle, $\angle P$, should your $\angle DRT$ be congruent to theirs? (Yes) Work with a classmate and test to see whether his angle and your angle seem to be congruent. Place your paper over a classmate's paper and hold them up to the light.

17. Choose a new unit angle smaller in size than a right angle. Then use your compass to construct an angle whose measure is 1. Call it $\angle ABC$. (Pupils will probably choose unit angles which are of different size.)

18. On the drawing you made for Exercise 17 construct with compass an angle whose measure is 2.

To state the measure of an angle we write:

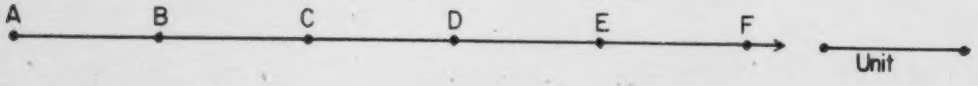
$m\angle DBA = 2$. The small "m" is read "measure of". We also write "$m\overline{AB} = 5$" to state the measure of a segment equals 5. "$m\overline{AB} = 5$" is read "the measure of $\overline{AB}$ equals 5."

Remember that a measure is a number.

Exercise Set 1

On a sheet of paper write the answers to the following exercises. Be sure to number each exercise.

1. State the measure of each segment named. The unit segment is shown at the right.



$\overline{AC}$, (m $\overline{AC}$ = 2) $\overline{AF}$, (m $\overline{AF}$ = 5) $\overline{BE}$, (m $\overline{BE}$ = 3) $\overline{DA}$, (m $\overline{DA}$ = 3) $\overline{FB}$, (m $\overline{FB}$ = 4) $\overline{CE}$. (m $\overline{CE}$ = 2)

Write your answer like this: m $\overline{AC}$ = 2.

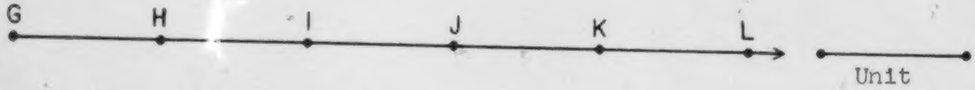
2. In the sketch below, name

   a) Four segments each of whose measure is 2.

   Write your answer like this: m $\overline{HJ}$ = 2.
   ($\overline{GI}$, $\overline{HJ}$, $\overline{IK}$, and $\overline{JL}$)

   b) Three segments each of whose measure is 3.
   ($\overline{GJ}$, $\overline{HK}$, and $\overline{IL}$)

   c) Two segments each of whose measure is 4.
   ($\overline{GK}$ and $\overline{HL}$)



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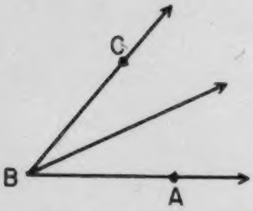
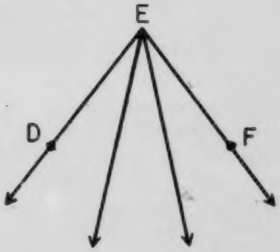
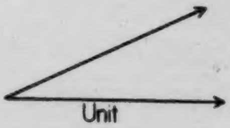
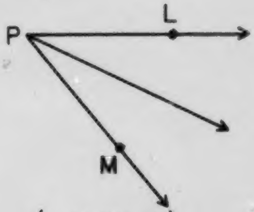
3. The small angles in the sketches are all congruent to the unit angle shown. State the measure of each of the angles named. Write your answer like this: m ∠ ABC = 2



(m∠ABC = 2)

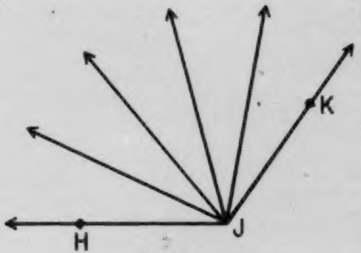






(m∠DEF = 3)

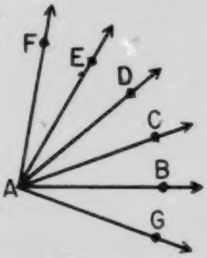
(m∠LPM = 2)



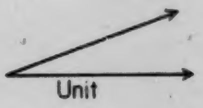
(m∠HJK = 5)

4. Each of the small angles in the sketch is congruent to the unit angle. State the measure of each angle named.

∠ GAC, ∠ BAE, ∠ CAF, ∠ DAG, ∠ BAC, ∠ FAG



m∠GAC = 2
m∠BAE = 3
m∠CAF = 3
m∠DAG = 3
m∠BAC = 1
m∠FAG = 5



5. Each of the small angles in each figure below is congruent to the unit angle. Using only the points which are marked, name:

a) An angle with measure 2. (∠HBI)

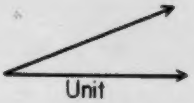
b) An angle with measure 4. (∠LDM)

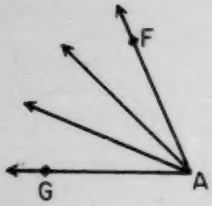
c) An angle with measure 7. (∠JCK)

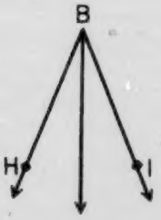
d) Two angles, the sum of whose measures is 7.
(∠FAG and ∠LDM or ∠HBI and ∠NEP)

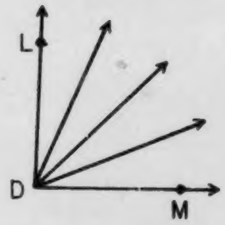
e) Two angles, the sum of whose measures is 9.
(∠NEP and ∠LDM or ∠JCK and ∠HBI)

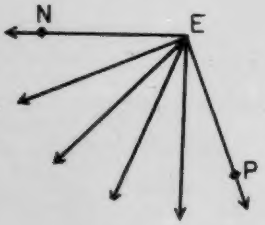
f) Three angles, the sum of whose measures is 16.
(∠JCK and ∠NEP and ∠LDM)

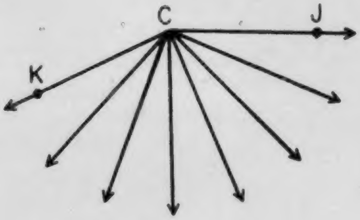












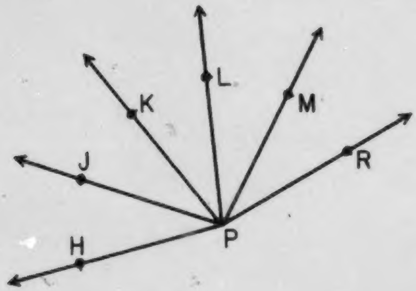
6. Each of the small angles in the figure below is congruent to the unit angle. Name:

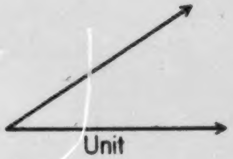
   a) Three angles with measure 2. (∠HPK, ∠JPL, ∠KPM, ∠LPR)

   b) Three angles with measure 3. (∠HPL, ∠JPM, ∠KPR)

   c) Two angles with measure 4. (∠HPM, ∠JPR)

   d) Four angles with measure 1. (∠HPJ, ∠JPK, ∠KPL, ∠LPM, ∠MPR)





7. In the figure of Exercise 6,

   a) m ∠ RPL = (2)   d) m ∠ MPK = (2)

   b) m ∠ LPH = (3)   e) m ∠ LPM = (1)

   c) m ∠ RPH = (5)   f) m ∠ MPH = (4)

   Look at your answers to Ex. 7a, b, and c.

8. Is this true? m ∠ RPL + m ∠ LPH = m ∠ RPH? (yes)

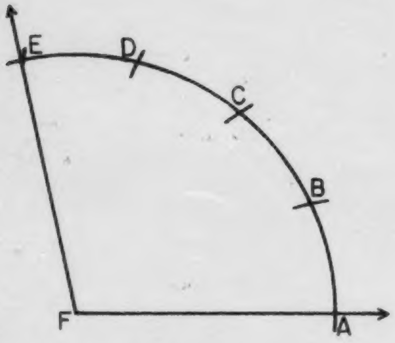
9. Now look at your answers for Exercises 7d, b, and f. Is this true? m ∠ MPK + m ∠ LPH = m ∠MPH? (No)

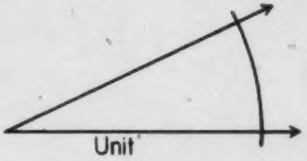
10. A boy wished to construct an angle of measure 4. He chose the unit angle shown below. He used his compass and straightedge to construct the ∠ AFE. A picture of his work is shown below. Look at the picture and answer the following:

a) What ray can you draw to complete an angle whose measure is 3? (Draw $\overrightarrow{FD}$ to complete ∠AFD whose measure is 3 or draw $\overrightarrow{FB}$ to complete an angle EFB whose measure is 3.)

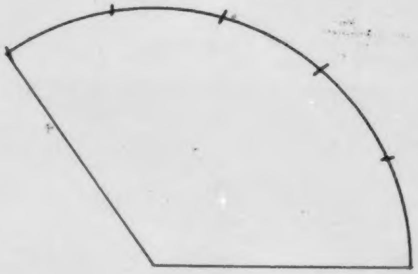
b) What ray can you draw to complete an angle whose measure is 1? (Draw one of the rays FD or FB to complete an angle whose measure is 1.)

c) Name the angle with measure 4. (∠EFA has a measure of 4)





11. Use the method shown in the sketch for Exercise 10 to construct an angle whose measure is 5. Use the unit angle of Exercise 10.



USE OF UNIT ANGLE IN MEASURING ANGLES

Objective: To develop the following understandings and skills:

1. We measure an angle by counting the number of congruent unit angles we may place successively in the interior of this given angle.
2. A compass may be used for laying off the unit angle in the interior of the given angle. The procedure is essentially that used in copying an angle.
3. At best, our measurements are approximations.
4. While $m \angle A = 4$, the size of $\angle A$ is 4 units.
5. If the size of an angle is closer to 7 units than it is to 6 units or 8 units, then we say $m \angle K = 7$ to the nearest unit.

Materials Needed:

Teacher: Chalkboard or string compass, straightedge

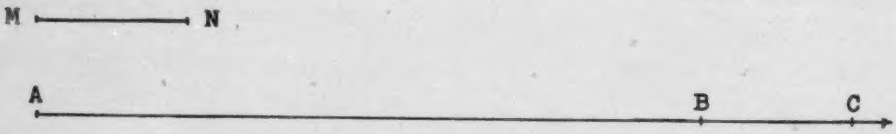
Pupil: Compass, straightedge

Vocabulary: To the nearest unit

USE OF UNIT ANGLE IN MEASURING ANGLES

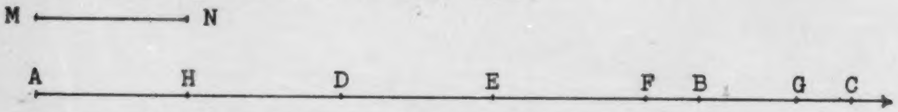
Exploration

You have used your compass to construct a line segment of a given measure and an angle of a given measure. Now suppose you wish to find the measure of $\overline{AB}$, using $\overline{MN}$ as unit. Trace $\overrightarrow{AC}$ and point B on a sheet of paper.



1. Now copy $\overline{MN}$ on $\overline{AB}$ with A as the left endpoint. Call the right endpoint, H. Repeat the process 4 more times to get line segments HD, DE, EF, and FG. Make each line segment congruent to $\overline{MN}$. How many such copies can you make on $\overline{AB}$? (4)

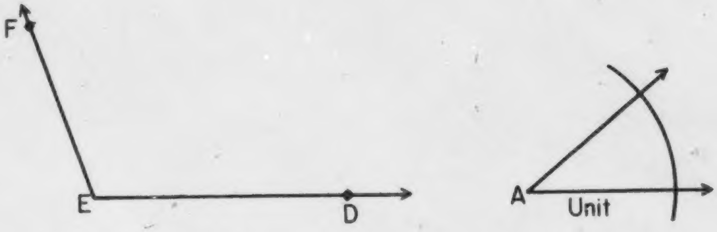
Your drawing should look like this:



In the sketch, $\overline{MN}$ was copied 4 times on $\overline{AB}$ so $m\ \overline{AF} = 4$. When $\overline{MN}$ is copied the last time, so $\overline{FG} \cong \overline{MN}$, you see that $m\ \overline{AG} = 5$.

Since point B is between point F and point G, $m\ \overline{AB} > 4$, and also $m\ \overline{AB} < 5$. Since B is nearer F than G, we say that $m\ \overline{AB} = 4$, to the nearest unit. If B were nearer G than F, then we would write $m\ \overline{AB} = 5$, to the nearest unit.

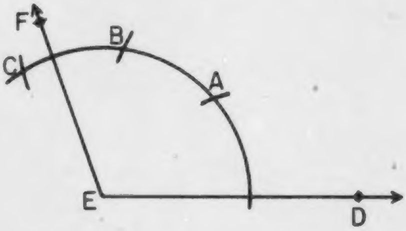
2. Suppose you are to find the measure of $\angle$ DEF, using $\angle$ A as unit.



Make a tracing of $\angle$ DEF and $\angle$ A on your paper. Can you use your tracing to estimate the measure of $\angle$ DEF? (yes)

m $\angle$ DEF = (3), to the nearest unit.

3. Now instead of tracing, use your compass as you did in Exercise 11, Set 1. Does your drawing look like this?



4. Draw $\overrightarrow{EA}$, $\overrightarrow{EB}$, and $\overrightarrow{EC}$.

Copy and complete the following statements.

5. m $\angle$ DEF > m $\angle$ (DEB), and

m $\angle$ DEF < m $\angle$ (DEC)

6. m $\angle$ DEB = (2), and

m $\angle$ DEC = (3)

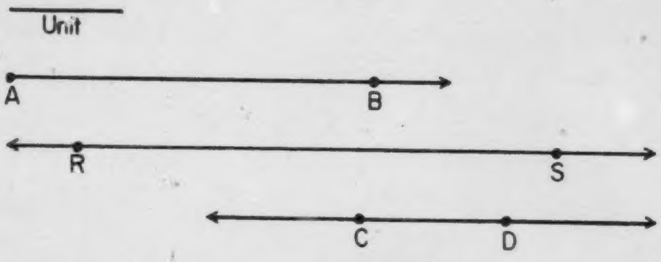
so m $\angle$ DEF > (2) and m $\angle$ DEF < (3)

7. m $\angle$ DEF is nearer (3) than (2).

m $\angle$ DEF = (3) to the nearest unit.

Exercise Set 2

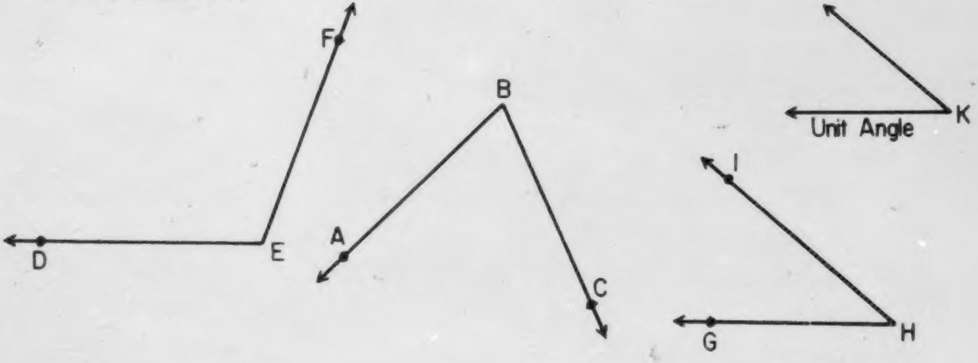
1. Make a copy of the following figures. Use the unit segment shown to find, to the nearest unit, the measure of each of the segments below. Use your compass.



Copy and complete the following statements

m $\overline{AB}$ = (3) m $\overline{RS}$ = (4) m $\overline{CD}$ = (1)

2. Trace the figures below on your paper. Use your compass and straightedge to find the measure, to the nearest unit, of each angle below. Use the unit angle K as the unit of measure.



m ∠ ABC = (2), to the nearest unit.

m ∠ DEF = (3), to the nearest unit.

m ∠ GHI = (1), to the nearest unit.

A SCALE FOR MEASURING ANGLES

Objective: To develop the following understandings and skills:

1. We can create a device for convenience in making measurements. The ruler is one such instrument.
2. The measuring device is marked with whole numbers in consecutive order, so that to each successive copy of the unit laid off on the device, there corresponds a number.
3. For convenience, we choose a unit angle which when laid off successively will fit into the half plane a whole number of times.
4. The instrument we will use for angle measure is a protractor. When the protractor is properly placed on the angle the measure of the angle can be read on the scale.
5. The two scales on a protractor are merely a convenience for measuring angles in either direction clockwise or counter-clockwise.

Materials Needed:

Teacher: Straightedge, chalkboard protractor, one half-disc, (semi-circular region), piece of tag board for each pupil to make an octon scale. A diameter of 4 inches is about right.

Pupil: Straightedge, protractor (to be made)

Vocabulary: Protractor, scale, octon

In order to carry through the development in items 1-7 of this Exploration the pupil will need his book open. Exercises 2, 3 and 4 call attention to three common errors in use of the protractor, namely

(a) failure to place the zero ray of the protractor along a ray of the angle to be measured,

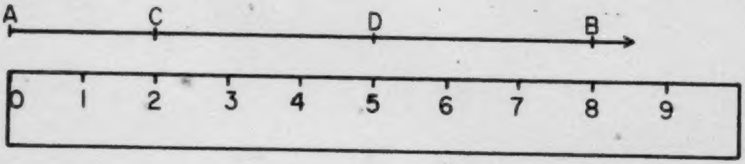
(b) failure to place the V mark of the protractor (intersection of rays) on the vertex of the angle to be measured.

(c) failure to read the correct scale.

A SCALE FOR MEASURING ANGLES

Exploration

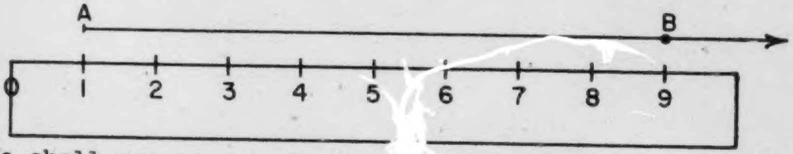
As you know, when you measure a line segment, you usually use a linear scale or ruler, with the endpoints of the unit segments marked with numerals. You place the ruler beside the segment and find the measure of the segment from the numerals on the ruler at the endpoints of the segment.



1. m $\overline{AC}$ = (2), m $\overline{AD}$ = (5), m $\overline{CD}$ = (3), m $\overline{AB}$ = (8)

2. Must you place the zero on the scale at the endpoint of the segment in order to find the measure of the segment?

(No, the zero on the scale does not have to be placed at the endpoint of the segment in order to find the measure of the segment. It could be placed like this, for example.



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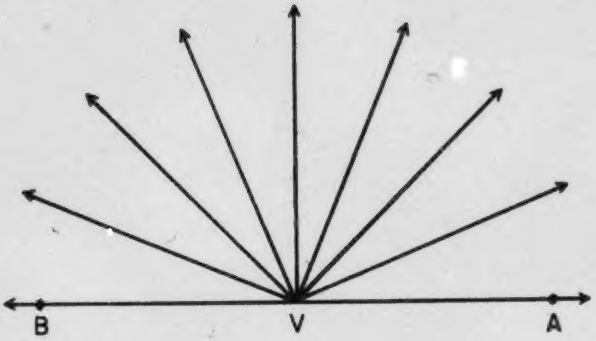
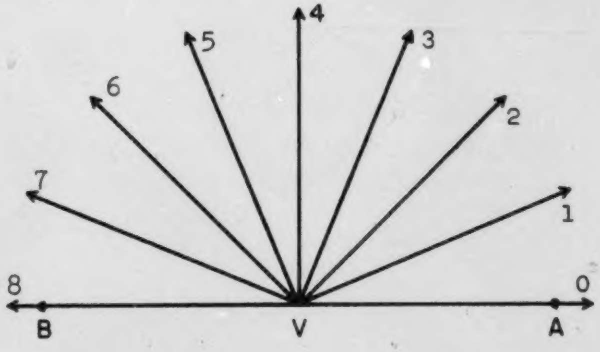
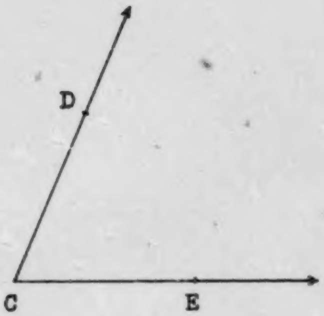
We shall use as a scale to measure angles a set of rays which are the sides of angles congruent to a unit angle. Any unit angle can be used but for convenience, we shall choose one so that eight of them with their interiors will exactly cover a half plane. We may name it whatever we want to. We will name our unit angle an "octon." Two of the rays, $\overrightarrow{VA}$ and $\overrightarrow{VB}$, are on the same straight line and extend in opposite directions from V.



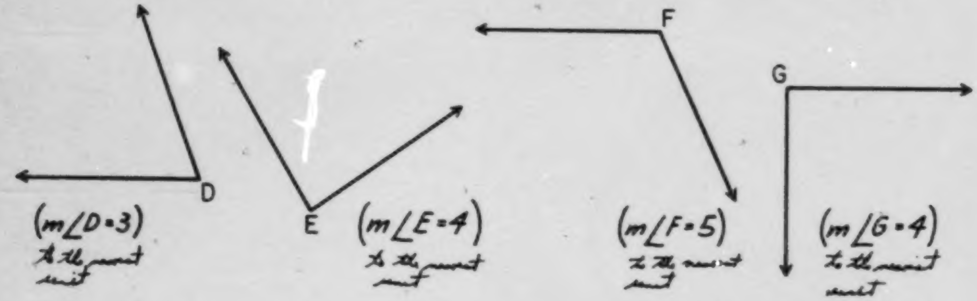
Then we will number the rays in order, putting 0 on the ray to the right ($\overrightarrow{VA}$) and ending the scale when we reach the ray on the same straight line as the zero ray ($\overrightarrow{VB}$).



3. Make a tracing of $\angle$ DCE. To measure $\angle$ DCE, how should the tracing be placed on the scale? Put C on V and $\overrightarrow{CE}$ on the zero ray. Then read the number of the ray on which $\overrightarrow{CD}$ falls, m $\angle$ DCE = (3).



4. Trace the angles below and use the angle scale to find the measure of each, to the nearest unit.


D
E
F
G
(m∠D = 3)
to the nearest unit
(m∠E = 4)
to the nearest unit
(m∠F = 5)
to the nearest unit
(m∠G = 4)
to the nearest unit

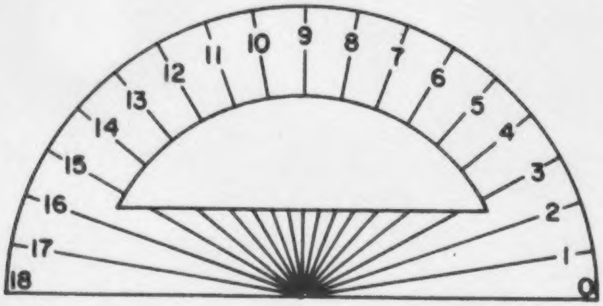
In measuring segments, it is convenient to have a linear scale marked off on a ruler. Then the ruler can be moved and placed beside a line segment.

In measuring angles, it is convenient to have an angle scale marked off on a protractor. Then the protractor can be moved and placed on an angle. Your teacher will show you how to make a protractor.

At this time there is value in a teacher demonstration lesson showing how to make a protractor and mark it off with an octon scale. The teacher should have a half disc of tag board about the size of the chalkboard protractor with the midpoint of the diameter of the disc clearly marked. On the chalkboard she should have a scale like the one on page 421 of the pupils' text. Two of the rays of the scale, VA and VC are on the same straight line and extend in opposite directions from V. Place the midpoint of the disc on V (the common endpoint of all the rays) on the scale of the chalkboard, making sure that the diameter of the disc falls along the rays VA and VC. The rays of the scale on the chalkboard should extend beyond the disc. Mark the point where each ray falls on the tag board disc. Then connect each of these points with the midpoint of the diameter of the disc.

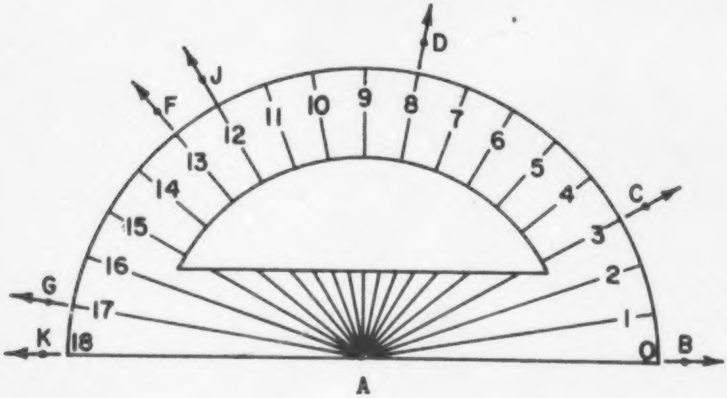
The teacher should have a tag board half disc about the size of a standard protractor, for each child. After the demonstration she can have each child make his own protractor, marked off in octons, using the scale of rays provided in his text on page 421.

Here is a picture of a cardboard protractor with a smaller unit angle than we used before. Only parts of the rays are shown.


0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18

The rays are broken because part of the cardboard is cut out so you can see the ray of the angle you are measuring.

5. Below is a sketch showing the protractor placed on a set of rays. The rays have the same endpoint, A, and the V-point of the protractor is on A . Find the measures of the angles named.


D
J
F
C
G
K
B
A
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18

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*Exploration

a) m $\angle$ BAC = (3)

b) m $\angle$ BAD = (8)

c) m $\angle$ BAJ = (12)

d) m $\angle$ BAF = (13)

e) m $\angle$ BAG = (17)

f) m $\angle$ GAF = (4)

g) m $\angle$ CAD = (5)

h) m $\angle$ DAF = (5)

i) m $\angle$ DAG = (9)

j) m $\angle$ CAJ = (9)

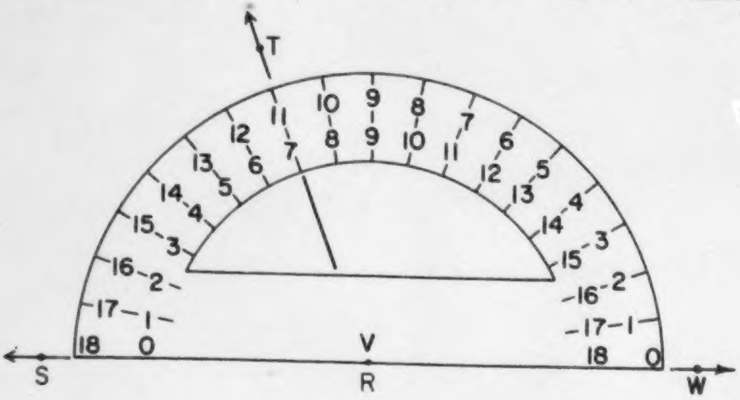
In addition to the scale with the zero ray at the right, many protractors also have another scale with the zero ray at the left. This scale is placed on the inner rim.

6. Look at the second scale on the protractor shown in the picture below. This scale is written on the inner rim. Zero is put on the ray to the left ($\overrightarrow{RS}$) and rays have been numbered in order until the ray on the same line with zero is reached ($\overrightarrow{RW}$). Write these numerals on the sketch you made for Exercise 5. Find the measures of the angles named in Exercise 5 using the new scale. Are the measures the same? (*The measures are the same.*)

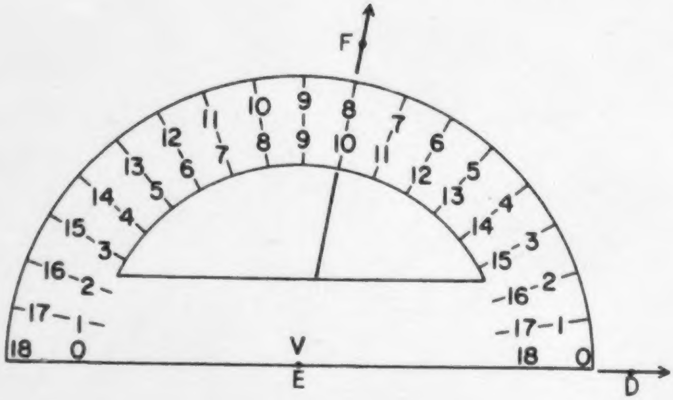
*This may be used as an independent exercise.

The advantages in having the two scales can be seen from the following sketches:

To measure ∠ SRT, the zero ray at the left is placed on $\overrightarrow{RS}$. You use the inner scale to find the measure. m ∠ SRT = (7)


T
18
17
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
V
S
R
W

To measure ∠ DEF, the zero ray at the right is placed on $\overrightarrow{ED}$. You use the outer scale to find the measure. m ∠ DEF = (8)


F
18
17
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
V
E
D

It is very easy to read the wrong scale by mistake. You will prevent most such errors by estimating the size of the angle as a check of your measurement. Of course, you can use either scale to measure the same angle, by moving the protractor.

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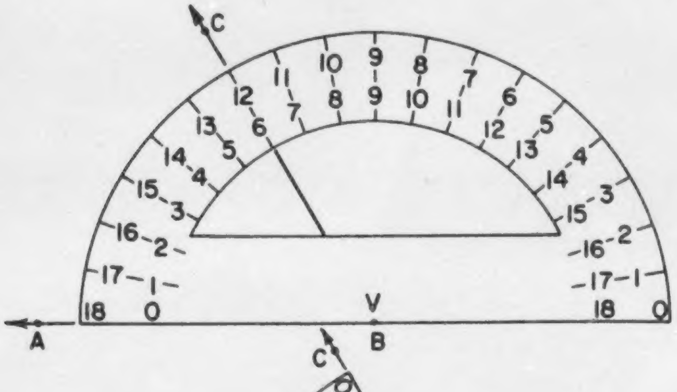
7. The following two sketches are copies of the same angle, ∠ ABC. In the first sketch, the protractor is placed so the zero ray on the left of the protractor is on ($\overrightarrow{BA}$). Which scale would you use to find the measure of the angle? (the inner scale) In the second sketch, the zero ray on the right of the protractor is placed on ($\overrightarrow{BC}$). Now which scale would you use to find the measure of the angle? (the outer scale) Is the measure of the angle the same either way? (yes)



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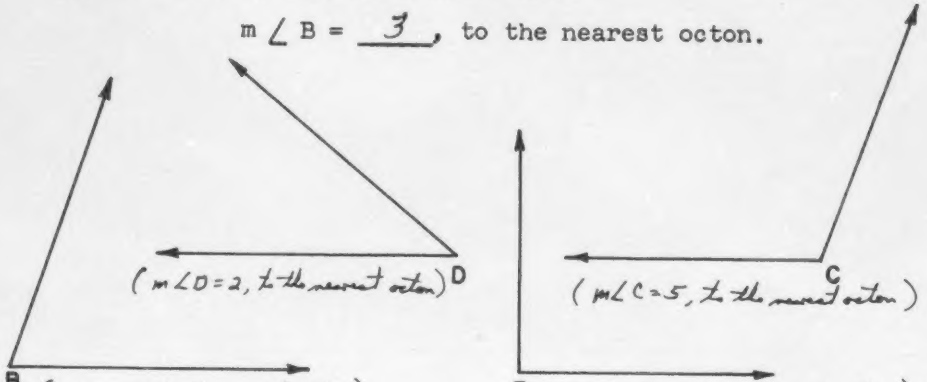
Exercise Set 3

1. Use the "octon" scale on your protractor to find the measure of each of the angles below (to the nearest octon). After you have measured an angle, check your measure by placing the protractor with a zero ray on the other side of the angle. Write your answer like this:

m ∠ B = 3, to the nearest octon.



(m∠D = 2, to the nearest octon)

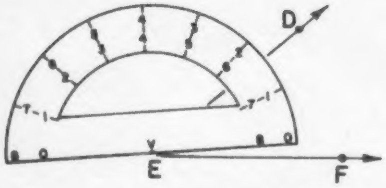
(m∠C = 5, to the nearest octon)

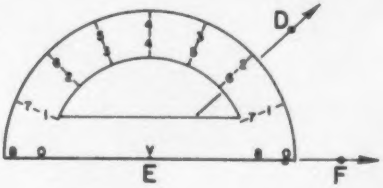
(m∠B = 3, to the nearest octon)

(m∠E = 4, to the nearest octon)

2. Which of these sketches shows the correct way to place the protractor to find the measure of ∠ DEF? Why?

m ∠ DEF = (2), to the nearest octon.



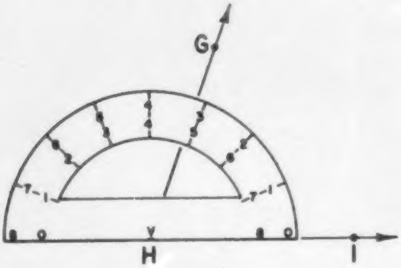


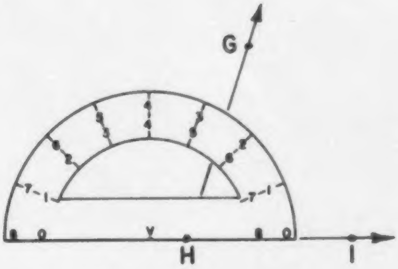
(The sketch on the right shows the correct way to place the protractor. The one on the left is tipped so that $\overrightarrow{EF}$ does not coincide with the zero ray.)

3. Which of these sketches shows the correct way to find the measure of $\angle$ GHI? Why?

m $\angle$ GHI = 3, to the nearest octon.

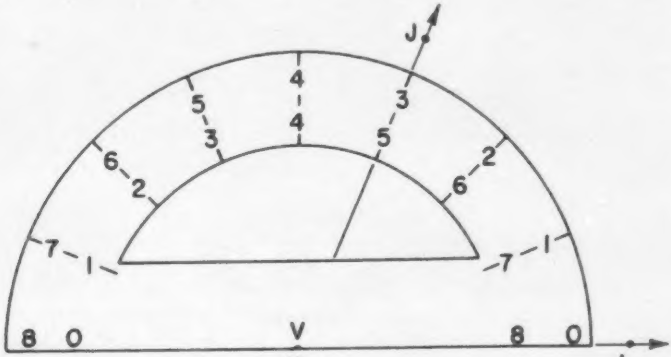
(The sketch on the left shows the correct way to find the measure of $\angle$GHI. In the sketch on the right, the V of the protractor is not placed on the vertex of the angle.)





4. A boy said that the measure of the $\angle$ JKL in octons is 5. What was his mistake? What is m $\angle$ JKL?

(The boy read from the wrong scale. He read from the inner scale instead of the outer scale. m$\angle$JKL = 3)



DRAWING AN ANGLE OF GIVEN MEASURE

Objective:	To develop skill in making drawings with a protractor of an angle whose measure is a given whole number of units.

Materials Needed:

Teacher:	Straightedge, octon protractor
Pupil:	Straightedge, octon protractor

Vocabulary:	No new words in this section.

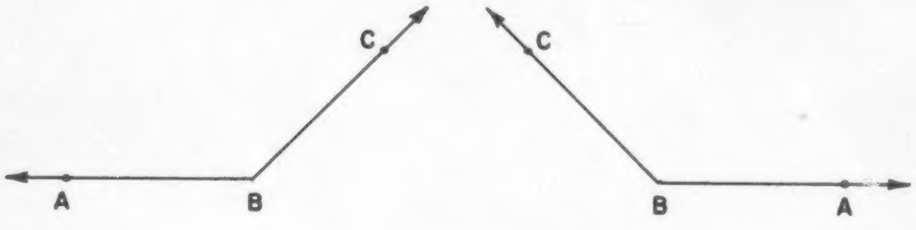
> The exploration is sufficiently detailed to be used as teaching procedure.

DRAWING AN ANGLE OF GIVEN MEASURE

Exploration

You can use your protractor to draw an angle whose measure, in octons, is to be a given whole number. Do you see how to use the protractor in this way?

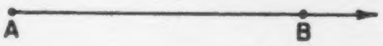
> Draw $\angle B$ so that $m\angle B$, in octons, is 6. Since the vertex must be point (B), draw $\overrightarrow{BA}$. Place the protractor with the V-point on the vertex and the zero ray of one scale on $\overrightarrow{BA}$. Mark a point C at the number 6 on the same scale. Remove the protractor and draw $\overrightarrow{BC}$. Each of these angles has a measure of 6, in octons. Does your angle look like one of them? (yes)



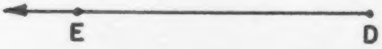
Exercise Set 4

In these exercises, draw rays and label points as in the sketches.

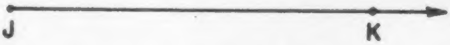
1. Copy the figure below on your paper.
   Draw on $\overrightarrow{AB}$ an angle with a measure of 5, in octons.
   Label it $\angle BAC$. Draw the angle so that $\overrightarrow{AC}$ is above $\overrightarrow{AB}$.



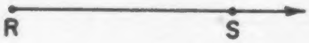
2. Copy the figure below on your paper. Draw an angle with a measure of 3 octons, using $\overrightarrow{DE}$ as one ray. Label it $\angle EDF$. Draw the angle so that $\overrightarrow{DF}$ is above $\overrightarrow{DE}$.



3. Copy $\overrightarrow{JK}$ on your paper. Draw an angle with a measure of 2 using $\overrightarrow{JK}$ as one ray. Label it $\angle KJL$. Draw the angle so that $\overrightarrow{JL}$ is below $\overrightarrow{JK}$.



4. Copy $\overrightarrow{RS}$ on your paper. Draw $\angle SRT$ whose measure is 7 using $\overrightarrow{RS}$ as one ray. Draw the angle so that $\overrightarrow{RT}$ is below $\overrightarrow{RS}$.



PRACTICE IN MEASURING ANGLES

Objective: To develop the following understandings and skills:

1. Not every angle has one ray drawn horizontally.
2. Regardless of the position of the rays of an angle, we use the same procedure to measure the angle with a protractor.
3. We may need to extend our representations of one ray of an angle in order to read its measure on the scale.
4. Extending our representation of the rays of an angle does not change the measure.

Materials Needed:

Teacher: Straightedge, octon protractor
Pupil: Straightedge, octon protractor

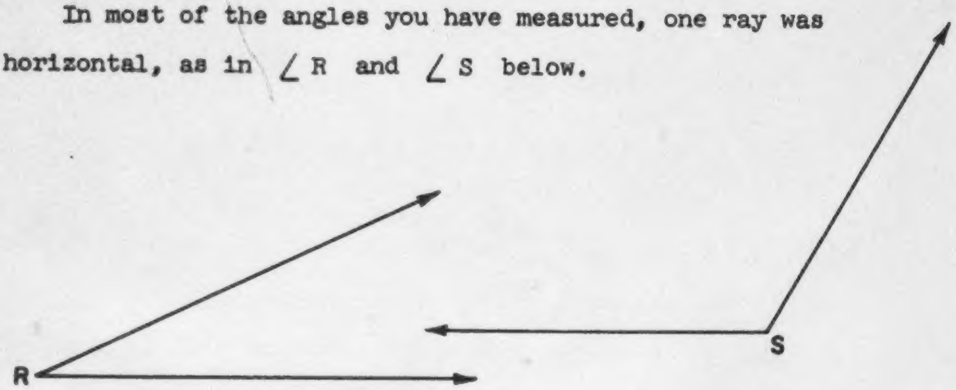
Vocabulary: No new words in this section

> If the exploration is followed closely all the understandings will be developed. The Exercises provide opportunity to practice the skills.

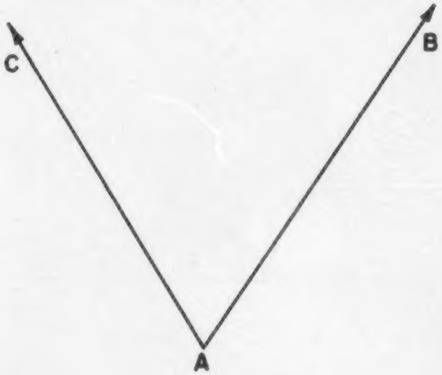
PRACTICE IN MEASURING ANGLES

Exploration

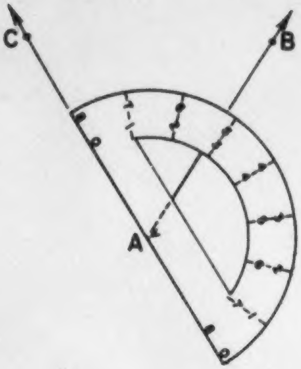
In most of the angles you have measured, one ray was horizontal, as in $\angle R$ and $\angle S$ below.

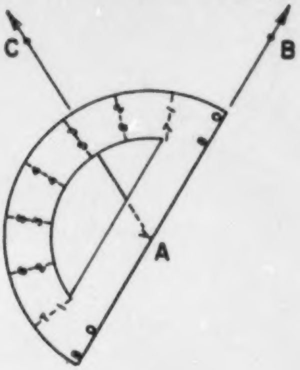


1. How would you find the measure of $\angle A$? This angle is in a different position from others you have measured. Its measure is found in the same way. Place your protractor so a zero ray falls on either $\overrightarrow{AB}$ or $\overrightarrow{AC}$. Be sure the V of the protractor is exactly on vertex A. The other ray of the angle can then be matched with the part of a ray marked on the protractor.



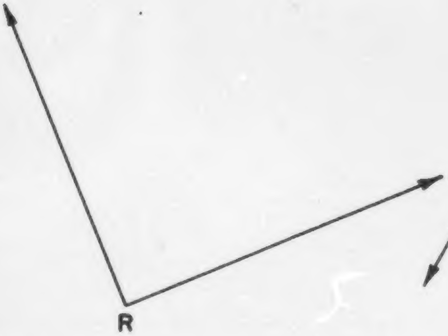
2. These sketches show the two ways to place the protractor.



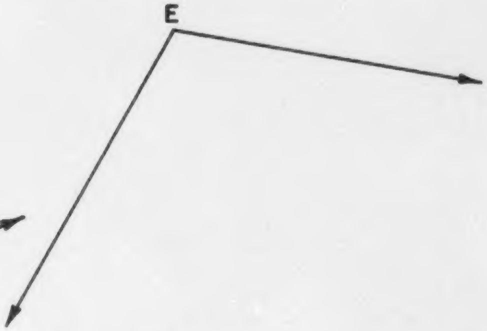


Put the zero ray on $\overrightarrow{AC}$ or put the zero ray on $\overrightarrow{AB}$. Does it make any difference in the measure whether the zero ray is on $\overrightarrow{AB}$ or on $\overrightarrow{AC}$? In each sketch, we see the measure of ∠ A to be about 3. Why is its measure 3 rather than 5, to the nearest octon? (It makes no difference if the zero ray is on $\overline{AB}$ or on $\overline{AC}$. The measure is 3 rather than 5 because the ray which falls under the protractor scale is nearest to 3 from the zero ray in each case.)

3. Find the measure of ∠ R and ∠ E.

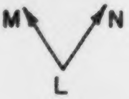


($m\angle R = 4$)

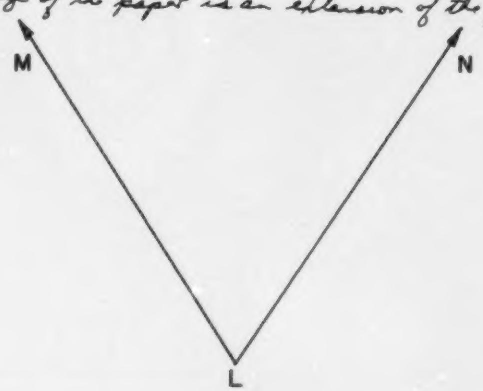


($m\angle E = 5$, to the nearest octon)

4. How do you think we can find the measure of $\angle L$? Can a protractor be placed on $\angle L$ so that you can read its measure? Do $\overrightarrow{LN}$ and $\overrightarrow{LM}$ have a definite length? (No)
(A protractor cannot be placed on $\angle L$ so that its measure can be read. It is hoped that the children will suggest that $\overrightarrow{LM}$ and $\overrightarrow{LN}$ can be extended so that $\angle L$ can be measured.)



5. If $\overrightarrow{LM}$ and $\overrightarrow{LN}$ are not long enough to extend beyond the protractor, can they be extended without changing the size of the angle? What is its measure? Check your measure by putting the zero-ray of your protractor on the other ray of the angle. Could you represent the rays in some way without drawing them?
(Clue: Try using a sheet of paper or some other kind of straightedge.) ($m\angle L = 3$, to the nearest octon. The rays could be represented without drawing them by placing a piece of paper on the ray so that the edge of the paper is an extension of the part of the ray shown.)



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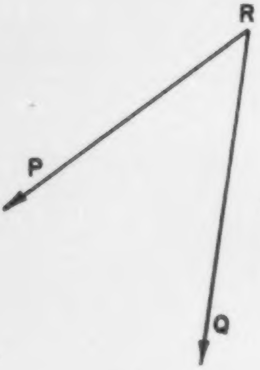
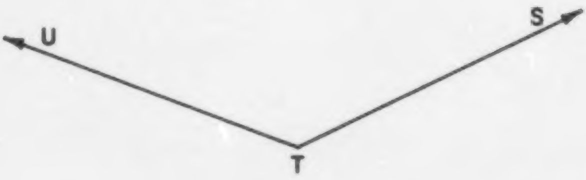
6. Find the measures of $\angle R$ and $\angle T$. You will need to show more of one or both rays.

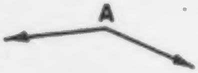


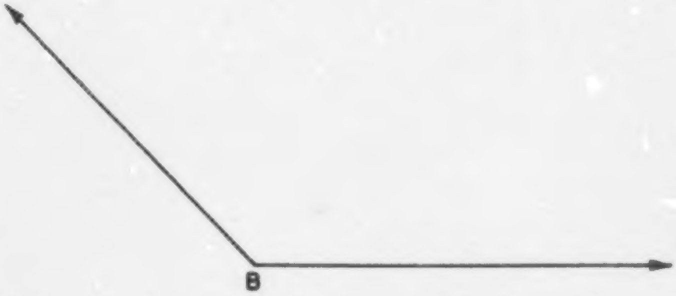
($m \angle R = 2$)



($m \angle T = 6$)

7. a) Which angle has the larger measure, $\angle A$ or $\angle B$? ($\angle A$)





b) Is the measure of $\angle A$ changed if you extend the part of its rays which are shown on this page? (no)

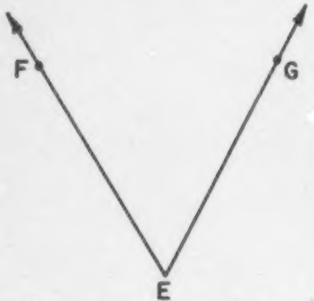
c) $m \angle A$ = (7), to the nearest octon.

d) $m \angle B$ = (6), to the nearest octon.

Exercise Set 5

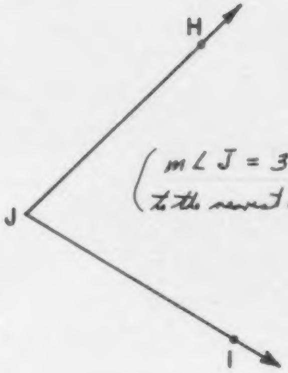
Find to the nearest octon the measures of the angles below. Use your octon protractor.

1.



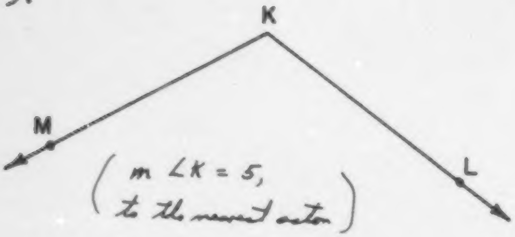
(m∠E = 3, to the nearest octon)

2.



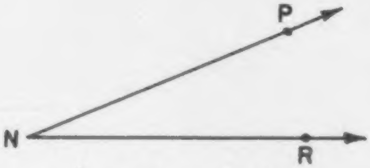
(m∠J = 3, to the nearest octon)

3.



(m∠K = 5, to the nearest octon)

4.



(m∠N = 1, to the nearest octon)

5. In Exercises 1 and 2, was m∠E = m∠J? (yes)
Is ∠E ≅ ∠J? (no) (Use a tracing)
Although ∠E is not congruent to ∠J, the measures, to the nearest octon, were the same number. If the unit angle were much smaller, then the fact that the angles are not congruent would be shown clearly in the measures.

A STANDARD UNIT FOR MEASURING ANGLES

Objective: To develop the following understandings and skills:

1. The desire to communicate gives rise to the need for a standard unit of measure.
2. The standard unit of angle measure is the degree. Its symbol is a raised "o", i.e. 42° is read 42 degrees.
3. One hundred eighty unit angles of 1 degree each may be laid off successively and represented on a semi-circular protractor.
4. The measure of an angle is a number. The size of an angle is given by naming its measure and also the unit used.
5. We concern ourselves only with angles whose measure in degrees are more than 0 and less than 180.

Materials Needed:

Teacher: Straightedge, standard protractor for chalkboard (with the numbers represented on the outer scale increasing in the counter-clockwise direction)

Pupil: Straightedge, standard protractor (if protractors are purchased, try to obtain ones in which the numbers represented on the outer scale increase in the counter-clockwise direction)

Vocabulary: Standard unit, size of an angle, degree.

‖ Follow the Exploration. ‖

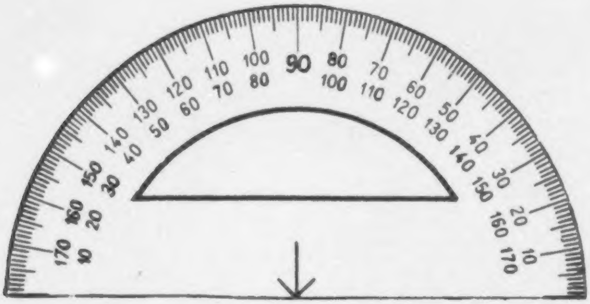
A STANDARD UNIT FOR MEASURING ANGLES

Exploration

As you know, the linear scale on a ruler is usually marked off using a standard unit such as the inch or the centimeter. A standard unit is one whose size has been determined by agreement among people. We would find it difficult to communicate with people or to carry on business if everyone made up his own units. What other standards of measure can you name?

There are also standard units for measuring angles, so that people throughout the world can communicate easily. The standard unit for measuring angles is the degree. The unit angle of one degree is smaller than the octon, the unit angle we used on the preceding pages. In fact, the octon is $22\frac{1}{2}$ times as large as an angle of one degree. Its measure in degrees is $22\frac{1}{2}$. The symbol for degree is $^{\circ}$. An angle of 15° means an angle whose measure, in degrees, is 15. As you work with your protractor you will discover that it takes 360 of these unit angles using a single point as a common vertex and their interiors, to cover the entire plane. Even in ancient Mesopotamia the angle of 1° was used as the angle of unit measure. The selection of their unit which could be fitted into a plane just 360 times was probably influenced by the fact that their year had 360 days.

1. Look at the side of your protractor on which the standard unit is the degree.



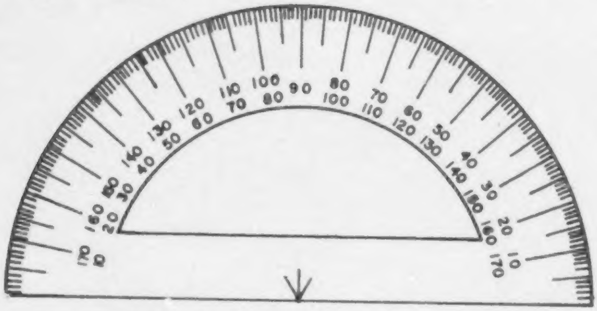
An angle of 1 degree is formed by rays, with endpoint V, through two of the marked points next to each other. Does this seem like a very small angle? (yes) Would it seem so small if the segment of the ray shown were extended to 15 feet?

(An angle of one degree would probably seem to be a small angle to the pupils. It would not seem so small if the segment of the ray shown were extended to 15 feet.)

2. Since 1 degree is so small, only every tenth degree is numbered on the scale. What other numbers are missing?
(all whole numbers from 0 to 180 except 10, 20, 30, . . . , 150, 160, and 170)

Why is 0 not printed on the scale? What is the largest number represented on the scale? Is its numeral printed? Why? (The numbers that are missing on the scale are 0 and 180. Zero is not printed on the scale because it marks a ray that coincides with the bottom edge of the protractor and, using our definition of an angle, we have no angle whose measure is exactly 0. The largest number represented is 180. Its numeral is not printed because it marks a ray that coincides with the bottom edge of the protractor, and, using our definition of an angle, we have no angle whose measure is exactly 180.)

3. Look at the side of the protractor on which the standard unit is the degree.



You use this standard protractor to measure an angle in degrees in the same way you used the scale on the other side to measure an angle in octons. You must be careful about the following things:

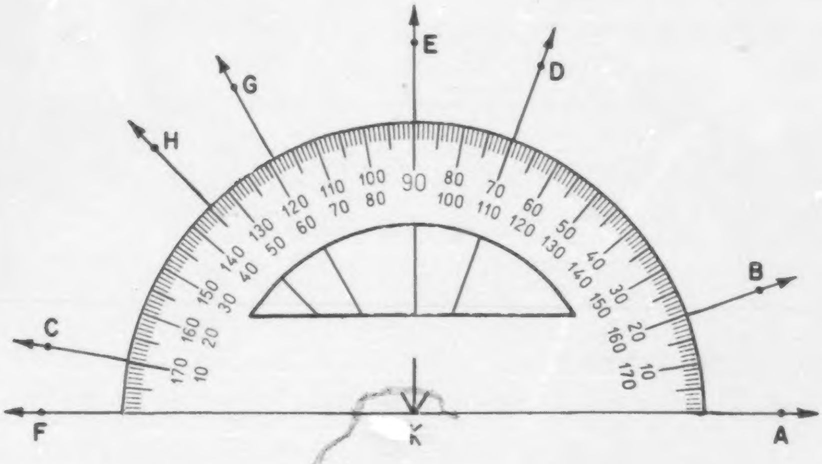
a. Place the V point of the protractor on the vertex of the angle. Be sure the protractor covers part of the interior of the angle.

b. Place the protractor with one of the zero rays exactly on one side of the angle. Notice whether this zero is a number on the inner scale or the outer scale. This is the scale you must use.

c. Find the point where the other side of the angle intersects the rim of the protractor. If not enough of the ray is shown to intersect the rim, can the rays of the angle be extended without changing the size of the angle? (Yes) Read the number at this point on the scale you chose in Step b.

Exercise Set 6

1. The sketch shows a protractor placed on a set of rays from point K. The V point of the protractor is on K. Find the measure, in degrees, of each angle named.



a) $m\angle AKB$ = (20)

b) $m\angle FKE$ = (90)

c) $m\angle AKC$ = (170)

d) $m\angle FKG$ = (61)

e) $m\angle AKD$ = (70)

f) $m\angle BKE$ = (70)

Imagine that the protractor has been moved so that the zero ray lies along $\overrightarrow{KB}$ (or $\overrightarrow{KE}$)

g) $m\angle CKD$ = (100)

Imagine that the protractor has been moved so that the zero ray lies along $\overrightarrow{KC}$ (or $\overrightarrow{KD}$)

h) $m\angle HKD$ = (65)

Imagine that the protractor has been moved so that the zero ray lies along $\overrightarrow{KH}$ (or $\overrightarrow{KD}$)

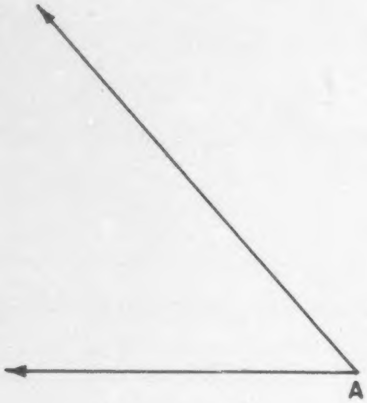
i) $m\angle DKB$ = (50)

Imagine that the protractor has been moved so that the zero ray lies along $\overrightarrow{KD}$ (or $\overrightarrow{KB}$)

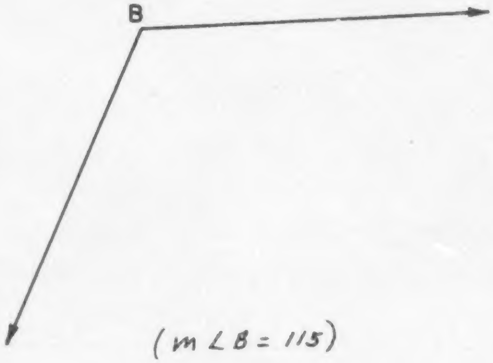
j) $m\angle HKC$ = (35)

Imagine that the protractor has been moved so that the zero ray lies along $\overrightarrow{KH}$ (or $\overrightarrow{KC}$)

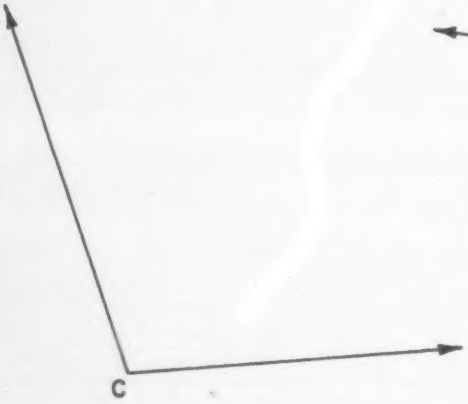
2. Use your protractor to find the measures, in degrees, of the following angles.



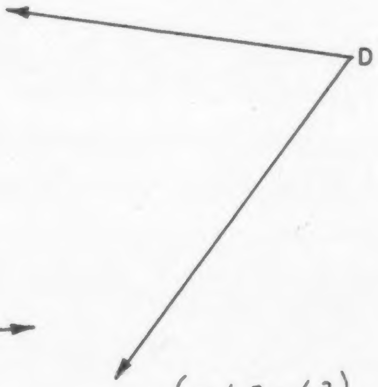
(m ∠A = 48)



(m ∠B = 115)



(m ∠C = 105)



(m ∠D = 62)

ESTIMATING THE MEASURE OF AN ANGLE

Objective: To develop the following understandings and skills:

1. Estimating the measure of an angle provides a check when reading the measure on a protractor scale.
2. Visualizing angles whose sizes are 45°, 90°, and 135° is helpful in estimating the size of a given angle.

Materials Needed:

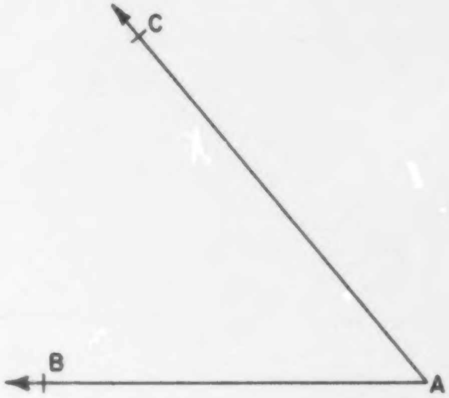
Teacher: Straightedge, chalkboard protractor
Pupil: Straightedge, standard protractor

Vocabulary: Estimate

The exploration in this section is very readable. The teacher might try having some group of pupils go through this Exploration as an independent activity and then check to be sure that understandings and skills are achieved. Keep in mind that one goal of this unit is to help the pupil improve his ability to read mathematics.

ESTIMATING THE MEASURE OF ANGLES

Exploration



Helen used her protractor to find the measure of $\angle A$. She made a mistake and read the wrong scale of her protractor, so she wrote for her answer $m \angle A = 130$. Max was asked to check her paper to see whether her answer was correct. Max said, "I do not have my protractor to find the measure of $\angle A$, but I know that Helen's answer is wrong." How did Max know that Helen's answer was not correct?

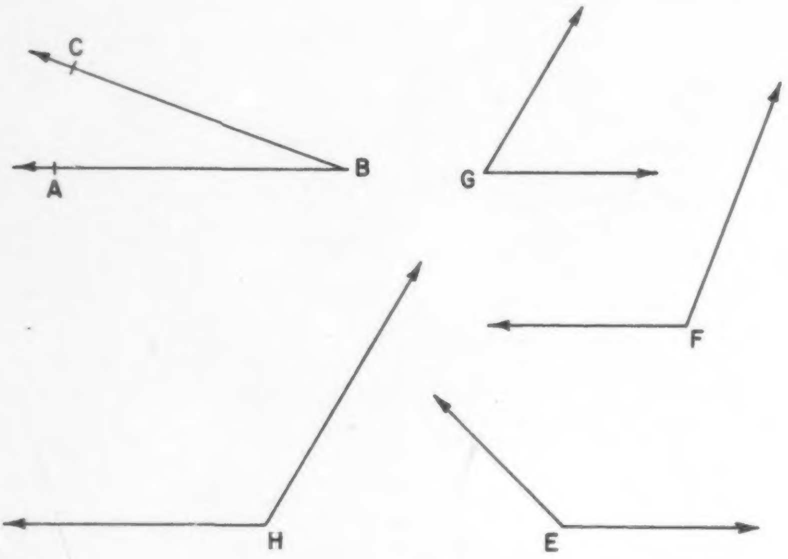
Whenever you can, you should make an estimate of an answer to a problem. Then if your answer is not close to this estimate you will suspect you may have made a mistake.

A good angle to use as a guide in estimating the measure of angles is a right angle.

1. Do you remember how to fold a paper to make a right angle? Just two folds are needed. (To make a model of a right angle, fold a sheet of paper once, in any way. Then fold again so that the fold lies on itself. The edges of the double fold and the angle fold make a model of a right angle.)

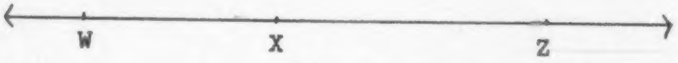
2. What is the measure of a right angle? Use your protractor if you need to. (The measure of a right angle, in degrees, is 90.)

3. Which of these angles has a measure greater than the measure of a right angle? Do not use your protractor. (For ∠ B, imagine $\overrightarrow{BD}$ which would make ∠ ABD a right angle. Place your pencil on the figure to represent $\overrightarrow{BD}$. Would $\overrightarrow{BD}$ be in the interior of ∠ ABC?) (No, therefore ∠B does not have a measure greater than that of a right angle.)
(∠E, ∠F, ∠H have measures greater than the measure of a right angle. $\overrightarrow{BD}$ would be in the exterior of ∠ABC.)



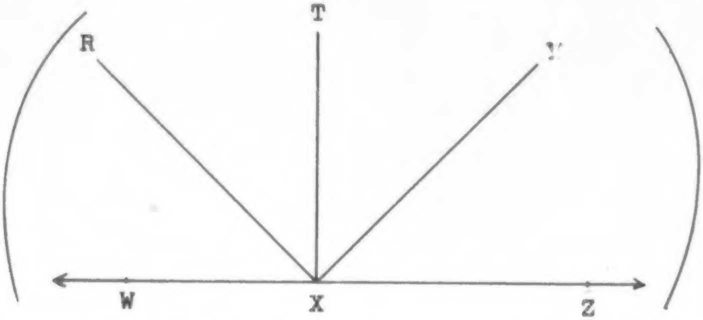
4. Which of the angles above have measures less than the measure of a right angle?
(∠B and ∠G have measures less than the measure of a right angle.)

5. Draw $\overleftrightarrow{WZ}$ on a piece of paper. On $\overleftrightarrow{WZ}$, choose a point X. Your drawing should look like this.



Use point X as vertex and $\overrightarrow{XZ}$ as one ray, and draw with your protractor an angle with a measure, in degrees, of 90. Call this $\angle$ ZXT.

Use X as vertex and $\overrightarrow{XZ}$ as one ray, and draw angles with measures, in degrees, of 45 and 135. Draw all three rays on the same side of $\overrightarrow{WZ}$. Label them so that m $\angle$ ZXY = 45 and m $\angle$ ZXR = 135.



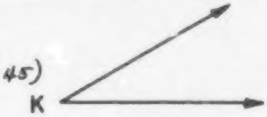
6. What other angles in the figure have a measure of 45? ($\angle YXT$, $\angle TXR$, $\angle RXW$)

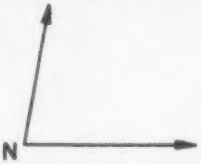
Name another angle which has a measure of 135. ($\angle$ WXY)

What other angles have a measure of 90? ($\angle$WXT and $\angle$ RXY)

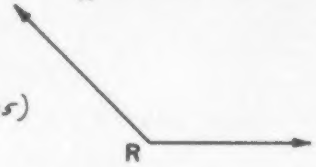
7. Look at each angle below and estimate its size. Use an angle of one degree as the unit. Now compare each angle with an angle in the drawing you made for Exercise 5.

a) Is m ∠ K nearer 0 or 45? (45)



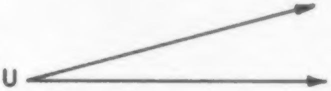


b) Is m ∠ N nearer 45 or 90? (90)

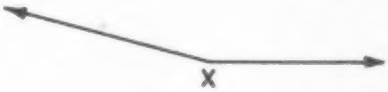


c) Is m ∠ R nearer 90 or 135? (135)

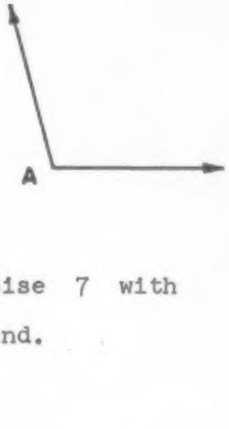
d) Is m ∠ U nearer 0 or 45? (0)



e) Is m ∠ X nearer 135 or 180? (180)



f) Is m ∠ A nearer 90 or 135? (90)



8. Now measure, in degrees, each angle in Exercise 7 with your protractor and write the measure you find.

a. m ∠ K = 30
b. m ∠ N = 80
c. m ∠ R = 135
d. m ∠ U = 15
e. m ∠ X = 165
f. m ∠ A = 105

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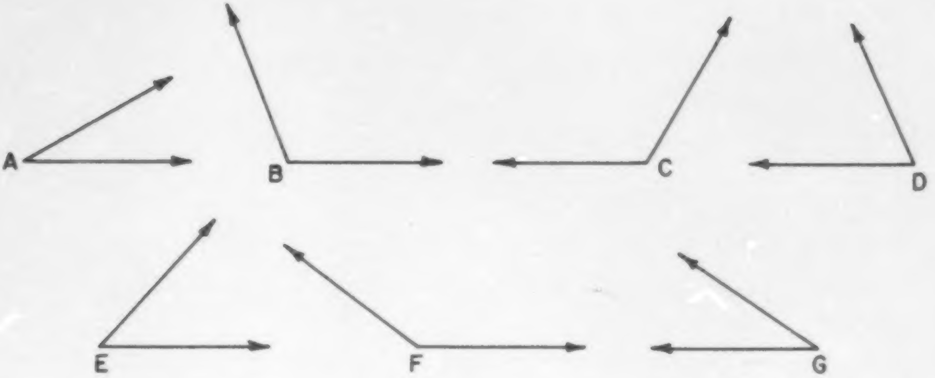
Exercise Set 7

1. Which of these angles has a measure less than the measure of a right angle? (∠A, ∠D, ∠E, ∠G)



2. Which of the angles above have a measure greater than 90? (∠B, ∠C, ∠F)

3. Look carefully at each angle. Choose the better estimate of its measure in degrees.

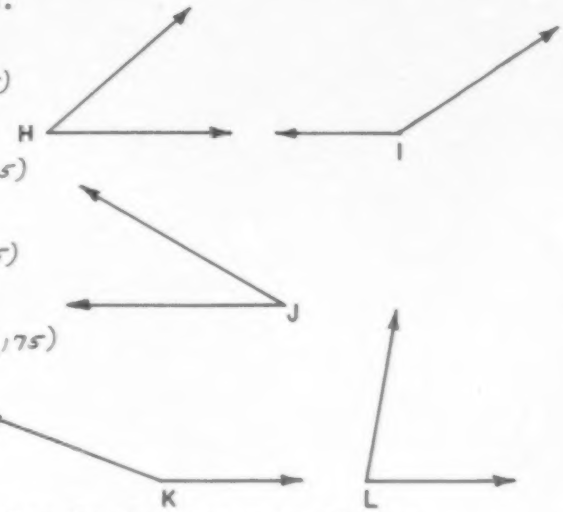
m∠H; 5 or 45 (45)

m∠I; 90 or 135 (135)

m∠J; 45 or 90 (45)

m∠K; 135 or 175 (175)

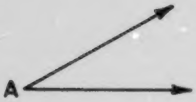
m∠L; 45 or 90 (90)

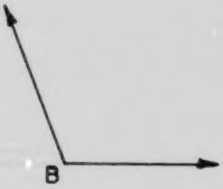


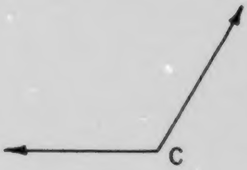
4. Measure each of the angles in Exercise 3 in degrees.
(m∠H = 40
m∠I = 146
m∠J = 30
m∠K = 160
m∠L = 80)

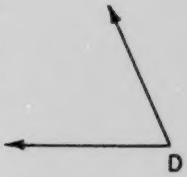
Exercise Set 7

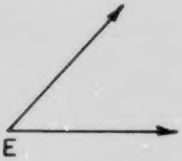
1. Which of these angles has a measure less than the measure of a right angle? (∠A, ∠D, ∠E, ∠G)

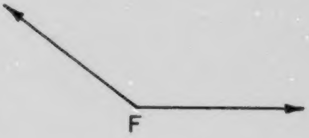


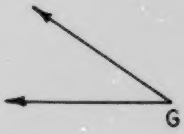












2. Which of the angles above have a measure greater than 90? (∠B, ∠C, ∠F)

3. Look carefully at each angle. Choose the better estimate of its measure in degrees.

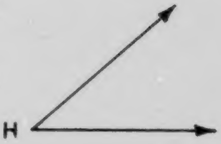
m∠H; 5 or 45 (45)

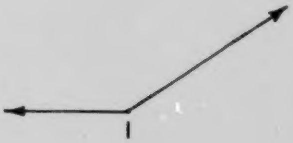
m∠I; 90 or 135 (135)

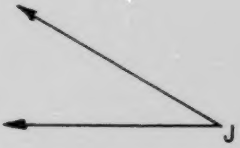
m∠J; 45 or 90 (45)

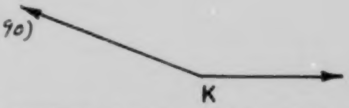
m∠K; 135 or 175 (175)

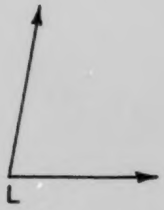
m∠L; 45 or 90 (90)







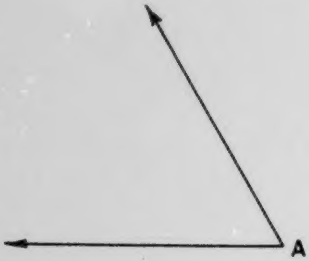


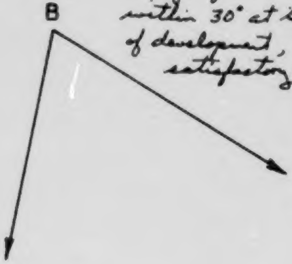


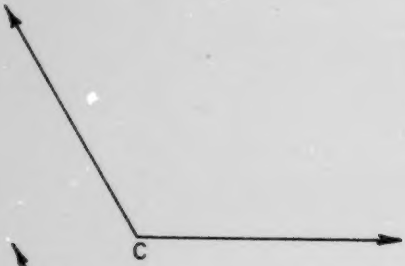
4. Measure each of the angles in Exercise 3 in degrees.

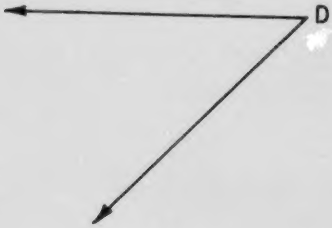
(m∠H = 40
m∠I = 146
m∠J = 30
m∠K = 160
m∠L = 80)

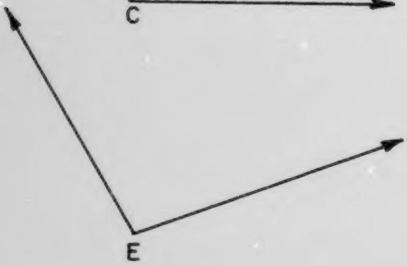
5. Estimate the measure of each angle in degrees. Write your answers like this: m ∠ A is about 60. (Pupils' answers will vary. If an estimate is within 30° at this stage of development, it is satisfactory.)

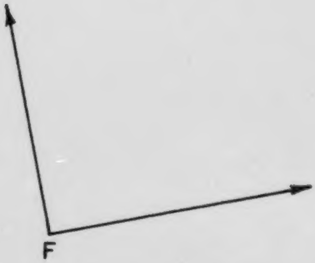


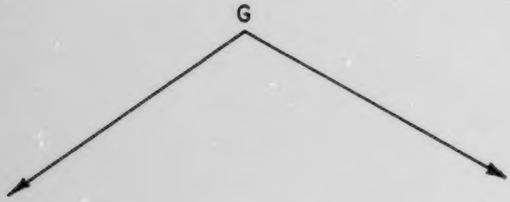


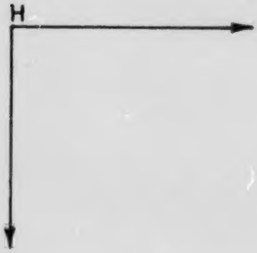












6. Measure, in degrees, each angle in Exercise 5 with your protractor.

(m ∠ A = 60
m ∠ B = 70
m ∠ C = 120
m ∠ D = 45
m ∠ E = 100
m ∠ F = 90
m ∠ G = 115
m ∠ H = 90)

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7. Use your straightedge to draw an angle which you think has the size given below. Then measure the angle with a protractor to see how closely the measures agree with your estimates.

	ANGLE	SIZE
a)	R	45°
b)	M	10°
c)	N	165°
d)	P	80°
e)	D	120°
f)	Q	178°

(Pupils answers will vary considerably. The goal is to give children practice in estimating the sizes of angles.)

SUM OF THE MEASURES OF ANGLES

Objective: To develop the following understandings and skills:

1. If two angles can be placed so that

   a. they have a common vertex
   b. they have a common ray
   c. their interiors do not intersect
   d. the other two rays of the angles lie on the same line,

   then the sum of the measures, in degrees, of the two angles is 180.

2. If, in addition to the four conditions above, the two angles are congruent, each has a measure, in degrees, of 90.

3. If a set of angles is laid off successively and the union of the angles and their interiors covers a half-plane and its edge, then the sum of the measures, in degrees, of these angles is 180.

Materials Needed:

Teacher: Straightedge, chalkboard protractor

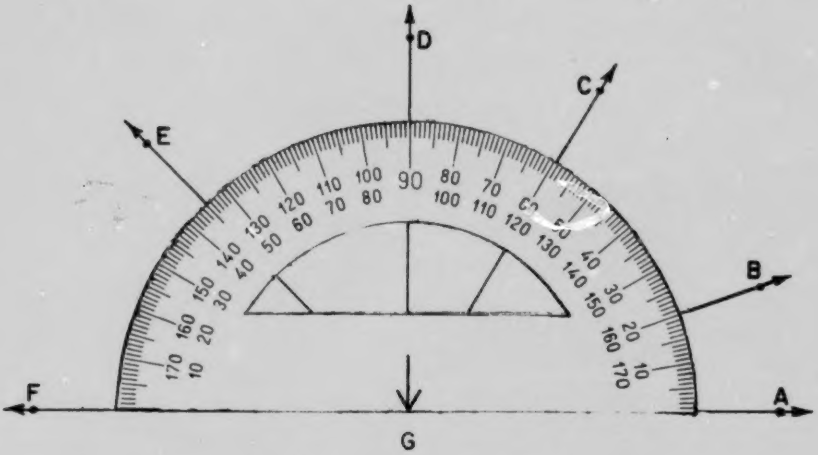
Pupil: Straightedge, protractor

Vocabulary: No new words in this section

> This exploration should be carried through with pupil texts open because the questions asked depend upon the drawing provided. The reading material in Exercises, Set 8 may well be a challenge to some pupils, because it is written largely in the symbolic language of mathematics. If necessary, the teacher might read through the Exercises with the pupils before they attempt them alone.

SUM OF THE MEASURES OF ANGLES

1. Copy and complete the following table. Find the measures of the angles from the sketch. (When the unit angle is not mentioned, use the measure in degrees.)


D
C
E
B
F
A
G
10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170

	NAME OF ANGLE	SIDES	MEASURES	
Pair 1	∠ AGB	$\overrightarrow{GA}$, $\overrightarrow{GB}$	(20)	Sum of measures (180)
	∠ BGF	$\overrightarrow{GF}$, $\overrightarrow{GB}$	(160)	
Pair 2	∠ AGC	($\overrightarrow{GA}$, $\overrightarrow{GC}$)	(60)	Sum of measures (180)
	∠ CGF	($\overrightarrow{GC}$, $\overrightarrow{GF}$)	(120)	

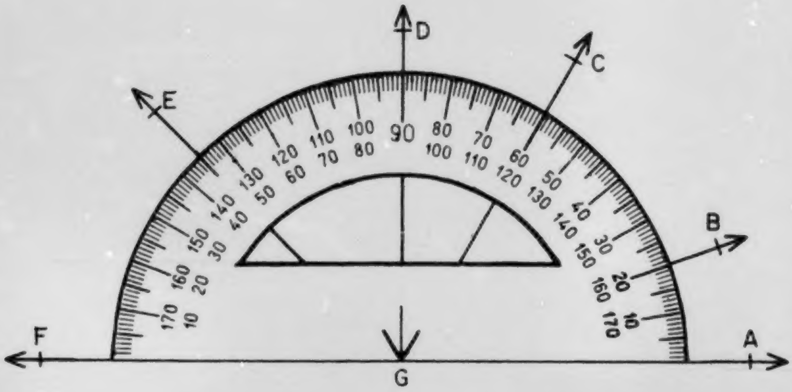
2. Trace with your finger the rays which form the angle pair, Pair 1, in Exercise 1.

   a) What ray is a side of both angles? $(\overrightarrow{GB})$

   b) What can you say about the other two rays? What is their intersection? What is their union? (The other two rays lie on the same straight line. Their intersection is Point G. Their union is $\overleftrightarrow{AF}$)

   c) Do the interiors of the angles intersect? (No)

   d) What is the sum of their measures? (180)



3. Trace with your finger the rays which form angle pair, Pair 2. Answer the questions as in Exercise 2 about this pair. Are your answers to questions b, c, and d the same as for Pair 1?

   (a. $\overrightarrow{GC}$ c. same as above
   b. same as above d. same as above)

4. Find a third pair of angles in the sketch for which the answers to questions b, c, and d are the same as for Pair 1 and Pair 2. Trace their rays with your finger.
(∠AGD and ∠DGF or ∠AGE and ∠EGF)

5. When these rays intersect at the same point and two of the rays form a line, what can you expect will be the sum of the measures of the two angles formed? (180)
(The pupils may not make a formal statement to this question but their responses should indicate the idea.)

6. List the names of all of the angles in the sketch whose interiors do not intersect. (There are five.)
(∠AGB, ∠BGC, ∠CGD, ∠DGE, ∠EGF)

7. Find the measure of each angle in your list.
(m∠AGB = 20, m∠BGC = 40, m∠CGD = 30, m∠DGE = 45, m∠EGF = 45

8. Find the sum of the measures of the five angles.
(The sum, in degrees, is 180.)

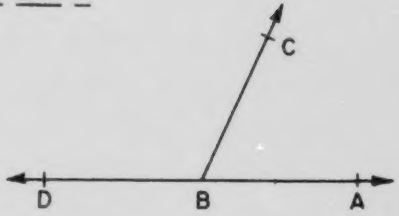
9. What conclusion can you reach from Exercise 6-8? (The idea being sought here is that if several rays are drawn from one point on a line and on the same side of the line, then the sum of all the angles whose interiors do not intersect is 180 in degrees.)

10. Name a pair of angles with $\overrightarrow{GF}$ a side of one angle, $\overrightarrow{GA}$ a side of the other, and $\overrightarrow{GD}$ a side of both angles.
(∠DGF and ∠AGD)

11. What is the measure of ∠AGD? ∠DGF? What kind of angle is ∠AGD? ∠DGF?
(m∠AGD = 90 ∠AGD is a right angle.
m∠DGF = 90 ∠DGF is a right angle.)

Exercise Set 8

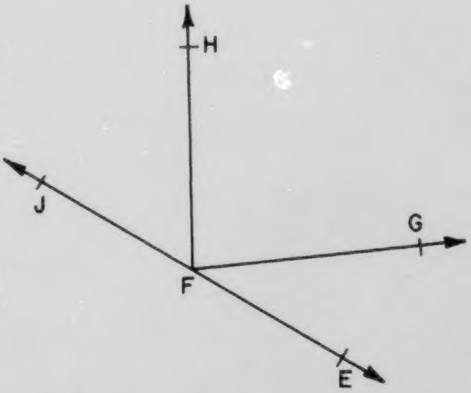
Use this figure for Exercises 1 and 2. $\overrightarrow{BA}$ and $\overrightarrow{BD}$ are on the same line.



1. If $\overleftrightarrow{DA}$ is a straight line, $m\angle ABC + m\angle CBD =$ (180)

2. If $m\angle ABC = 65$, then $m\angle CBD =$ (115)

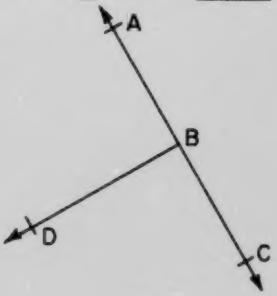
Use this figure for Exercises 3 and 4. $\overrightarrow{FE}$ and $\overrightarrow{FJ}$ are on the same line.



3. Is it true that $m\angle EFH + m\angle JFG + m\angle HFG = 180$? (no)
   If not, what true statement can you make?
   ($m\angle EFG + m\angle GFH + m\angle HFJ = 180$)

4. If $m\angle JFH = 58$ and $m\angle EFG = 36$, then $m\angle HFG =$ (86)

5. If $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are on a straight line, and $\angle ABD \cong \angle DBC$, then $m\angle ABD =$ (90) and $m\angle DBC =$ (90).
   $\angle ABD$ and $\angle DBC$ are (right) angles.



SUMMARY

1. If three rays have the same endpoint, and two of the rays form a line, then the sum of the measures, in degrees, of the two angles formed is 180.

2. If several rays are drawn from a point on a line, all on the same side of the line, the sum of the measures, in degrees, of all the angles formed whose interiors do not intersect is 180.

3. If one ray is drawn from a point on a line and the two angles formed are congruent, each angle is a right angle and its measure, in degrees, is 90.

Chapter 8

AREA

PURPOSE OF UNIT

This unit is one of a series intended to constitute the study of measurement in the elementary grades. It is intended that the study of measurement as it applies to line segments, angles, area, and volume provide the child with a valuable experience in a branch of geometry which is a powerful tool in the physical world. These units form a continuous and coordinated treatment of the topic of measurement. The sequence of topics in each unit parallels the historical development of this body of knowledge. The Teachers' Commentaries for the units on Linear Measurement and Measurement of Angles contain discussions on (1) the difference between comparisons of the sizes of sets of discrete objects and comparisons of the sizes of sets of points which form continuous curves, (2) the need for intuitive awareness of comparisons of size between sets of points which form continuous curves, before any formal procedure for measurement is set up, (3) the concept of an arbitrary unit of measure, and the need to select as a unit a thing of the same sort as the thing whose measure we wish to find (a line segment as a unit for measuring length, a region as a unit for measuring area, etc.) (4) the creation of a scale for convenience, and (5) the final step of selecting a standard unit to meet the needs of our society.

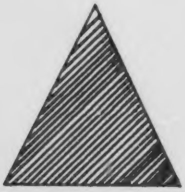
MATHEMATICAL BACKGROUND

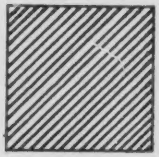
Comparing Areas

Let us recall how the subject of linear measurement was approached in Chapter 9, Grade 4, since area will be approached in a similar manner in the present unit. First we encountered the intuitive concept of *comparative length* for line segments: any two line segments can be compared to see whether the first of them is of *smaller length*, or the *same length*, or *greater length* than the second. Corresponding to this we have in the present unit the idea of *comparative area* for plane regions. (Recall that by definition a plane region is the union of a simple closed curve and its interior.) Even when they are rather complicated in shape, two regions can, in principle at least, be compared to see whether the first of them is of *smaller area*, or the *same area*, or *greater area* than the second.

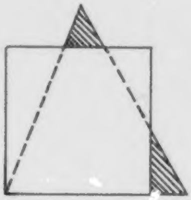
In the case of line segments, this comparison is conceptually very simple: we think of the two segments to be compared, say $\overline{AB}$ and $\overline{CD}$, as being placed one on top of the other in such a manner that A and C coincide: then either B is between C and D, or B coincides with D, or B is beyond D from C, etc. This conceptual comparison of line segments is also easy to carry out approximately using approximate physical models (drawings and tracings, etc.) of the line segments involved.

In the case of plane regions, this comparison is more complicated both conceptually and in practice. This is because the *shapes* of the two plane regions to be compared may be such that neither will "fit into" the other. How, for example, do we compare in size (area) the two plane regions pictured below?

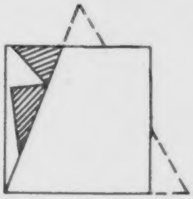




If we think of these regions as placed one on top of the other, neither of them will fit into the other.



In this particular case, however, we can think of the two pieces of the triangular region which are shown shaded in the figure above as snipped off and fitted into the square region thus:

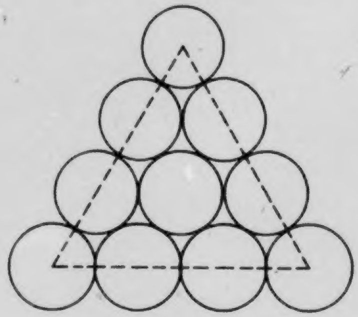


This shows that the triangular region is of smaller area than the square region. Early in this unit, the pupil is asked to carry out approximately some simple comparisons of this sort, using paper models of the plane regions to be compared. The pupil actually cuts up the paper model of one region and fits the pieces, without overlapping, on the model of the other region. Here we rely on the pupil's intuition to "see" that a plane region can be thought of as "cut up into pieces" and even "reassembled" to form a figure of different shape, without changing its area. (See the "robot exercise" in the pupils' book.) As the figures involved become more complicated in shape, this sort of comparison becomes increasingly difficult in practice. We need a better way of estimating the area of a region.

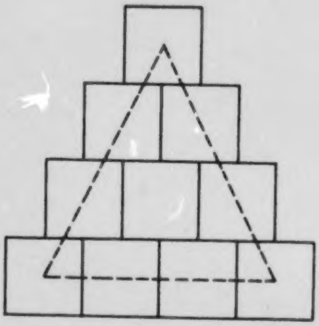
Units of Area

Let us recall what we did next in the case of linear measurement. We chose a unit of length. That is, we selected a certain line segment and agreed to consider its length to be described or measured, exactly, by the number 1. In terms of this unit we could then conceive of line segments of lengths exactly 2 units, 3 units, 4 units, etc., as being constructed by laying off this unit successively along a line 2 times, 3 times, 4 times, etc. The process of laying off the unit successively along a given line segment also yielded (under-and-over-) estimates for the length of the given line segment in terms of the unit. For example, the length of a given line segment might have turned out to be greater than 3 units (underestimate) but less than 4 units (overestimate).

We now proceed similarly in the measurement of area. The first step is to choose a unit of area, that is, a region whose area we shall agree is measured exactly by the number 1. Regions of many shapes as well as many sizes might be considered. An important thing about a line segment as a unit of length was that enough unit line segments placed end to end (so that they touch but do not overlap) would together cover any given line segment. Similarly, we need a unit plane region such that enough of them placed so that they touch but do not overlap will together cover any given plane region. Circular regions do not in general have this property. For example, if we try to cover a triangular region with small non-overlapping congruent circular regions, there are always parts of the triangular region left uncovered.



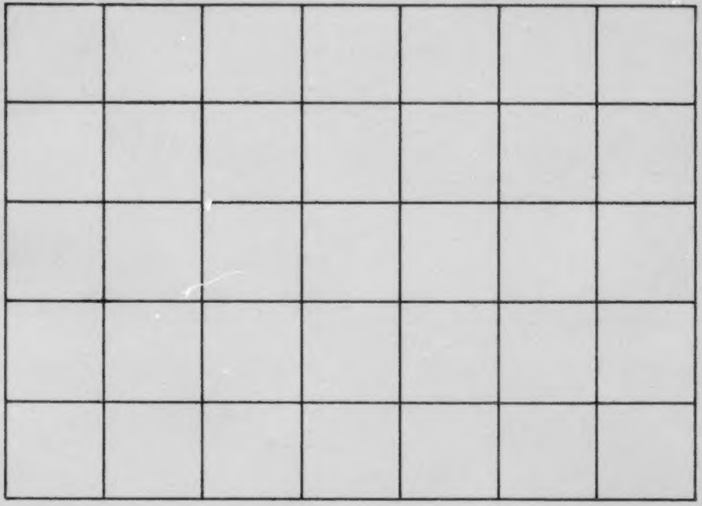
On the other hand we can always completely cover a triangular region, or any region, by using enough non-overlapping congruent square regions.



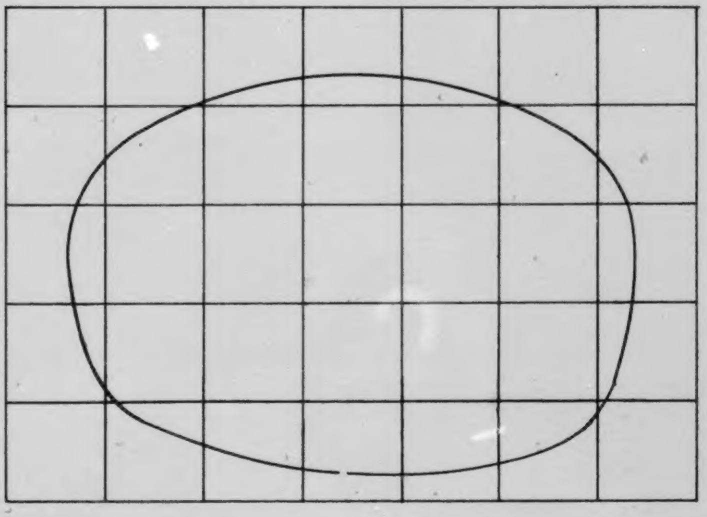
While a square region is not the only kind of region with this covering property, it has the advantage of being a simply shaped region whose size can be conveniently chosen by letting its side be of length 1 unit. More importantly, it then turns out that the use of such a square region as the unit of area makes it easy to compute the area of a rectangle by forming the product of the numbers measuring the lengths of its sides, a matter which we shall discuss in the unit.

Estimating Areas

Just as we used the unit of length to find underestimates and overestimates for the length of a given line segment, so we can now use the unit of area to find underestimates and overestimates for the area of a given region. A convenient tool for this purpose is a regular arrangement or grid of square unit regions as shown on the following page.

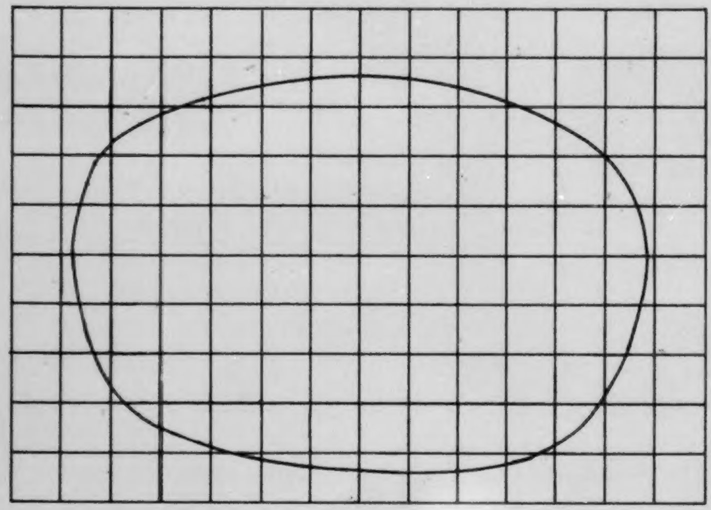


To use such a grid in estimating the area of a given region, we think of it as superimposed on the region. This is illustrated below for an oval region.



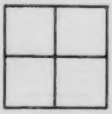
We can verify by counting that 12 of the unit regions pictured are contained entirely in the given oval region. This shows that the area of this region is at least 12 units. In fact, we can see that the area is more than 12 units. We can also verify by counting that there are 20 additional unit regions pictured which together cover the rest of the region. Thus, the entire region is covered by 12 + 20 or 32 units. This shows that the area of this region is at most 32 units. In fact, we can see that the area is less than 32 units. That is, we now know that the area of the region is somewhere between 12 units and 32 units.

In Chapter 9, Grade 4, on Linear Measurement, we saw that more accurate estimates of lengths could be achieved by using a smaller unit. The same is true with area. To illustrate this fact, let us re-estimate the area of the same oval region considered on the previous page, using this time the unit of area determined by a unit of length just half as long as before.



As before, we can verify by counting that there are 59 of the new unit regions pictured which are contained entirely in the given oval region. This shows that the area of the region is at least 59 (new) units. We can also verify by counting that there are 37 additional unit regions pictured which together cover the rest of the region. Thus, the entire region is covered by 59 + 37 or 96 of the new units. This shows that the area of this region is at most 96 (new) units. That is, we now know that the area of the region is somewhere between 59 (new) units and 96 (new) units.

Let us compare these new estimates of the area with the old ones. Each old unit contains exactly 4 of the new units, as is clear from the figure below.



Thus, the old estimate of 12 units becomes 4 x 12, or 48, new units; and the old estimate of 32 units becomes 4 x 32, or 128, new units. Thus, in terms of the new unit, the old estimates tell us that the area of the region lies somewhere between 48 units and 132 units, whereas the new estimates tell us that this area lies somewhere between 59 units and 96 units. Plainly, the new estimates based on the smaller unit are the more accurate ones.

In principle it would be possible to estimate the area of region of quite general shape to any desired degree of accuracy by using a grid of sufficiently small units in this way. In practice, the counting involved would quickly become very tedious. Furthermore, where approximate drawings are used to represent the region and grid involved, we would, of course, also be limited by the accuracy of these drawings.

Actually, the emphasis here is not so much on accurate estimates as it is on simply leading youngsters to grasp the following basic sequence of ideas.

1. Area is a feature of a region (and not of its boundary).

2. Regions can be compared in area (smaller, same, greater), and regions of different shapes may have the same area.

3. Like a length, an area should be describable or measurable, exactly, by some appropriate number (not necessarily a whole number).

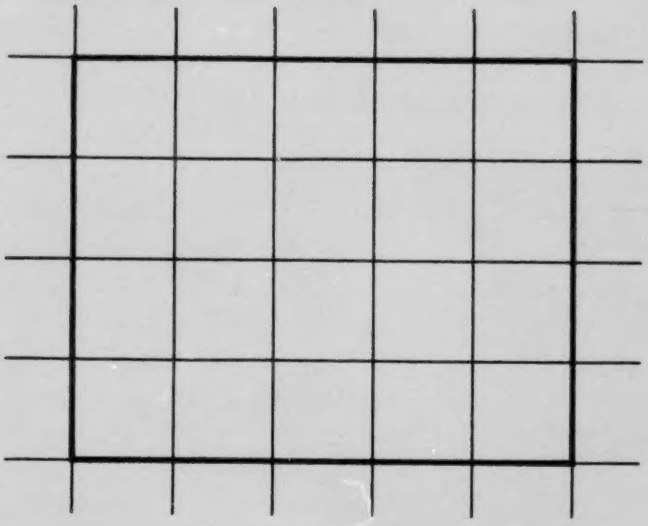
4. For this purpose, we need to have chosen a unit of area just as we earlier needed a unit of length.

5. The number of units which measures exactly the area of a region can be estimated approximately, from below and from above, by whole numbers of units.

6. In general, smaller units yield more precise estimates of an area.

Computing the Area of a Rectangular Region

First we consider the case of a rectangle whose length and width are given exactly by whole numbers of units of length; for example, a rectangle of length exactly 5 units and width exactly 4 units. The region bounded by such a rectangle would naturally be placed on a grid of unit regions thus, where we are using as unit of area a square region whose side is of unit length.



We see at once by counting that its area is exactly 20 units.

In this Unit, we lead pupils to the observation that for such a rectangular region "area = length × width." (Strictly speaking, we do not multiply lengths; we multiply numbers only. So we understand the formula "area = length × width" to be an abbreviation for the assertion that the number of units measuring the area is the product of the numbers of units of length measuring respectively the length and the width.) Note that the given rectangular region has 4 rows of 5 units each, which suggests the equation

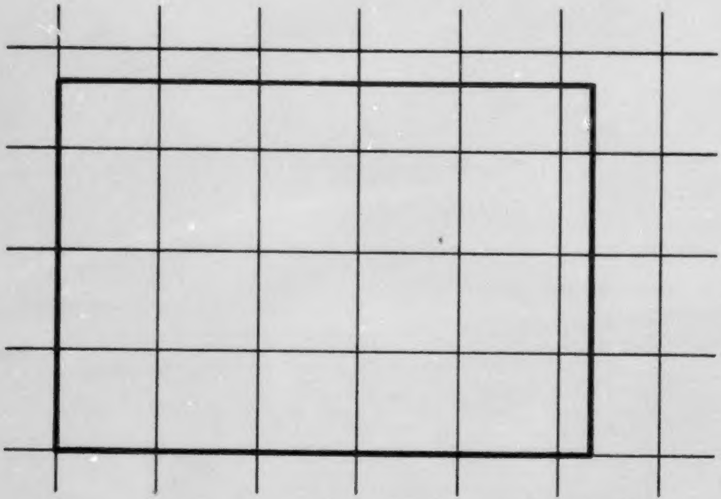
$$4 \times 5 = 20.$$

We ask the pupils to think of differently shaped rectangular regions of this same area and lead them to note the corresponding equations

$$2 \times 10 = 20$$

$$1 \times 20 = 20.$$

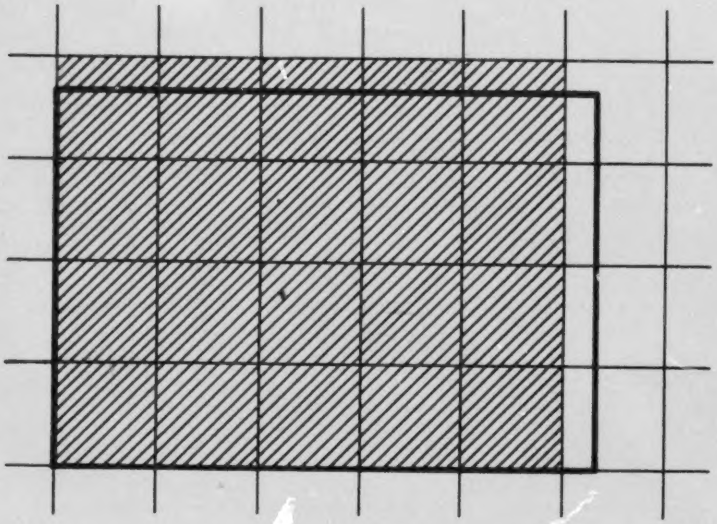
Although it is not treated in the pupil's book, the teacher might wish to extend the treatment to include the case of a rectangle whose length and width are measured only "to the nearest unit"; for example, a rectangle of length slightly more than 5 units and width nearly 4 units, as shown below.



By counting unit regions in the superimposed grid (just as in our earlier example of the egg-shaped region) we see that an underestimate for the area of the region bounded by this rectangle is 15 units, and that an overestimate for this area is 24 units. Now the actual length of the rectangle lies between 5 and 6 units of length and is 5 units, to the nearest unit. The width of the rectangle lies between 3 and 4 units of length and is 4 units, to the nearest unit. Therefore, if in the formula

$$\text{area} = \text{length} \times \text{width}$$

we use length and width as measured to the nearest unit, we obtain 4 × 5, or 20, units of area. This is exactly the area of a slightly different rectangular region, namely the one shown shaded in the figure below.

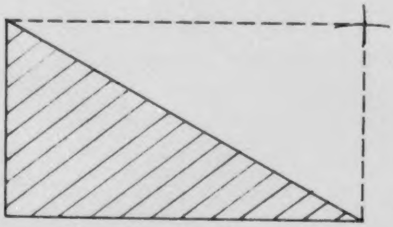


It is plain from this figure that the shaded rectangular region contains the same unit regions, and is covered by the same unit regions, as the original rectangular region (whose rectangular boundary is shown in heavy outline in the figure). This illustrates the fact that when we apply the formula

$$\text{area} = \text{length} \times \text{width}$$

to a rectangular region whose length and width are measured only to the nearest unit, we do not in general get the exact area of this region. What we do get is an estimate for this area. This estimate (20 units of area in the example) is a reasonalbe one in that it necessarily lies between our underestimate (15 units) and our overestimate (24 units) for this area.

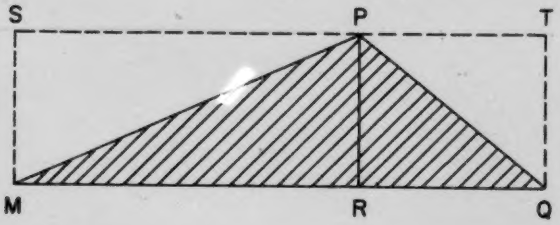
Once we have learned to calculate the area of a rectangular region, we can easily calculate the area of a right-triangular region. Given a right triangle, we first locate the fourth vertex of the rectangle whose other three vertices are the vertices of the triangle. (The pupil learns to do this approximately.) using a drawing of the triangle and a compass.)



The resulting rectangular region is seen to consist of two congruent non-overlapping triangular regions. The measure of each triangular region is one-half the measure of the rectangular region.

Now the measure of the rectangular region is the product of the measures of two of its adjacent sides. It follows that the measure of the right-triangular region is one-half the product of the measures of the sides forming the right angle. When one of these sides is taken as an altitude of the right triangle, the other becomes the base. In these terms the measure of the right-triangular region is one-half the product of the measures of its base and its altitude.

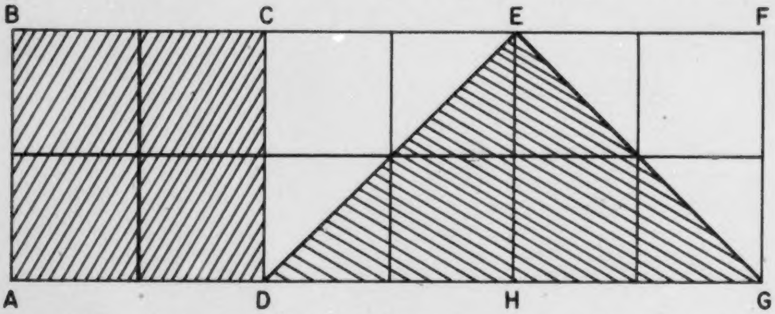
This is extended to the more general case of a triangle MPQ with altitude $\overline{PR}$ and base $\overline{MQ}$, as pictured below.



As a review, the argument presented in the pupils' book makes use of associative and distributive laws from arithmetic. Alternatively this can be argued as outlined below.

1. The measure of rectangular region MQTS is the product of the measures of $\overline{MQ}$ and $\overline{PR}$.
2. The measure of triangular region MPQ is one-half the measure of rectangular region MQTS.
3. Therefore, the measure of triangular region MPQ is one-half the product of the measure of its base $\overline{MQ}$ and the measure of its altitude $\overline{PR}$.

An alternate figure, which holds only for the isosceles triangular region, is sometimes useful in clarifying the ideas. It can be used with the children as a different apprroach after they have worked through the material given.



Observe that the measure of rectangle ABCD is the same as the measure of $\triangle$DEG, whereas, $m\ \overline{HE} = m\ \overline{AB}$ and $m\ \overline{DG} = 2\ m\ \overline{AD}$.

The charts on the next three pages tabulate concepts of measurement in connection with length and area. The first two pages of this tabulation have already appeared in Chapter 9, Grade 4, on Linear Measurement. A fourth page, concerning Volume, will be added in the appropriate later unit.

CONCEPT

The unit for measuring must be of the same nature as the thing to be measured: a line segment as a unit for measuring line segments, an angle as a unit for measuring angles, etc. For convenience in communication, standard units (foot, meter, degree, square foot, square meter, etc.) are used.

The measure of a geometric object (line segment, angle, plane region, space region) in terms of a unit is the number (not necessarily a whole number) of times the unit will fit into the object.

Measurements yield underestimates and overestimates of measure in terms of whole numbers of units. In the case of line segments and angles, they also yield approximations to the nearest whole number of units.

Segments and regions can be thought of as mathematical models of physical objects. Physical terms are used to describe the physical objects and the physical terms are also used in discussing mathematical models. This is acceptable provided the correct mathematical interpretation of the physical terms is understood.

A curve in space may have length.

Some measures of a figure may be calculated from other measures of that figure.

A set consisting of disjoint segments (several separate pieces) may also have the property of length.

A line segment is a set of points consisting of two different points A and B and all points between A and B on the line containing A and B. Sometimes we say "segment" when it is clear that we mean "line segment."

We use a line segment as a unit for measuring line segments.

We use the word "meter" to name the segment which is accepted as the standard unit for linear measurement. We use "inch," "foot," and "yard" to name certain other units which are defined with relation to the standard unit.

The measure of a line segment in terms of a unit is the number (not necessarily a whole number) of times the unit will fit into the line segment. The unit segments may have common endpoints but must not overlap.

In measuring a line segment, as the unit becomes smaller, the interval within which the approximate length may vary, decreases in size. The precision of a measurement depends upon the size of this interval. The smaller the unit, the smaller the interval and the more precise the measurement.

The length of a line segment in terms of a given unit consists of (1) the measure of this segment in terms of this unit together with (2) the unit used. Example: if the measure (in inches) of a line segment is 5, then its length is 5 inches.

Many of the familiar curves in a plane or in space also have length. We can bend a wire to the shape of the curve and then straighten the wire to represent a segment.

We calculate the perimeter of a triangle or other polygon. If the measures of the sides of a triangle (where the unit of measurement is the inch) are 4, 5, and 6, then the perimeter of the triangle is measured by the number $4 + 5 + 6$ or 15. We say that the perimeter of the triangle is 15 inches.

A figure consisting of several segments that do not touch may have length. The measure of the figure in terms of a given unit is the sum of the measures of the separate segments in terms of that unit.

AREA

The union of a simple closed curve and its interior is a plane region. Examples are a triangle and its interior or a circle and its interior.

A plane region is used as a unit for measuring plane regions.

We use the phrase "square meter" to name a plane region which is accepted as the standard unit. The sides of the square boundary of this region are standard unit segments (the meter). The phrases "square inch," "square foot," "square yard" are used to name square regions where the sides of the boundary are the "inch," "foot," and "yard," respectively.

The measure of a plane region in terms of a unit is the number (not necessarily a whole number) of times the unit will fit into the plane region. The unit regions may have parts of their boundaries in common but must not overlap.

As with linear measurement, the precision of a measurement of a plane region depends upon the size of the region used as a unit. The smaller the unit, the more precise the measurement.

The area of a plane region in terms of a given unit consists of (1) The measure of this region in terms of this unit together with (2) the unit used. Example: if the measure (in square inches) of a region is 6, then its area is 6 square inches.

The surfaces of many of the familiar solid regions have area. For many of these we can take a plane region and cut it into pieces so that the pieces will cover the surface.

We can calculate the measure of a rectangular region. If the measures of the adjacent sides of the rectangle (where the unit of measurement is the inch) are 2 and 3, then the measure of the region is 2 × 3 or 6. The area of the plane region is 6 sq. in.

The concept of plane region may be extended to some plane figures other than a simple closed curve and its interior. For example, the figure consisting of two triangular regions that do not touch is a plane region and its measure is the sum of the measures of the triangular regions.

TEACHING THE UNIT

Each section of the chapter is divided into Explorations and Exercises. It is intended that, unless otherwise indicated in the teacher's commentary, the Exploration be a teacher-directed activity with the fullest possible pupil participation. Each teacher will decide whether the Exploration can best be directed with opened or closed books. If the books are closed, you may or may not wish to go over the Exploration again when the books are opened. The pupil book contains the Exploration as a written record of the activity in which the class has engaged. The pupils should work independently on the Exercises. Since the Exercises serve not only for maintenance and drill, but also are sometimes developmental in character, it is suggested that class discussion of the Exercises follow their completion by the pupils. The answers which are included in the commentary may prove helpful in these discussions.

It is recommended that throughout this unit, wherever pertinent, the teacher have the pupils shade or color the plane regions they will use. In this way, the pupil will not always just see a simple closed curve when he is really working with a plane region.

WHAT IS AREA?

Objective: To develop the following understandings and skills:

1. Sometimes we compare the sizes of objects by comparing their lengths.

2. Sometimes we compare the sizes of objects by comparing the sizes of flat surfaces they cover.

3. The size of a representation of a bounded flat surface does not change when we bend or fold it.

4. To establish a unit for measuring a bounded flat surface, we need the concept that a simple closed curve separates the plane into three sets of points: the set of points of the curve, the set of points of the interior of the curve, and the set of points of the exterior of the curve.

5. The union of a simple closed curve and its interior is called a plane region. This is just one kind of region.

6. We use a plane region as a unit for measuring plane regions.

7. The area of a plane region is the measure of the plane region and the unit used to make the measure. We measure a plane region to find its area.

Materials:

Teacher: Any objects considered necessary for discussion of gross comparison of size in the first Exploration.

Pupils: Crayon, any objects necessary for answers to Exercise Sets 1 and 2.

Vocabulary: Surface, bounded, flat surface, plane region, area, triangular region, polygonal (pȯ liğ' ȯ năl) region, rectangular region.

> By the end of this section, the pupil should be able to differentiate between a figure which has length and a figure which has area.

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Chapter 8

AREA

WHAT IS AREA?

Comparing Sizes of Regions

Exploration

You have had experience in comparing the sizes of line segments and the sizes of angles. Look around your classroom. Find representations of two line segments which are not the same length. Can you tell without making any measurement which is longer? Find representations of two angles, which are not the same size. Can you tell which is larger without using the compass or protractor?

Exercise Set 1

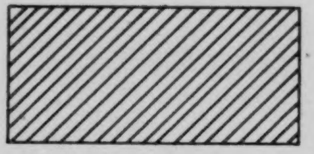
In each of the following, tell which is larger:

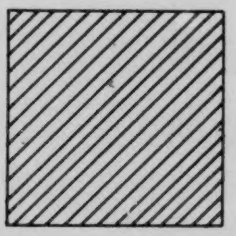
1. A sheet of typing paper or a stamp. (a sheet of typing paper)
2. A pin head or a dinner plate. (a dinner plate)
3. A pillow case or a bed sheet. (a bed sheet)
4. A television screen or a motion picture screen. (a motion picture screen)
5. A nickel or a dime. (a nickel)
6. A wash cloth or a handkerchief. (depends on object)
7. A window or its window-shade. (If "window" means the pane, the shade should be larger.)
8. Your classroom floor or your classroom ceiling. (probably the same)
9. The sole of your shoe or the sole of your friend's shoe. (depends on objects)
10. A sheet of your notebook paper or this page of your text. (depends on objects.)
11. Did you know the answer to the above exercises immediately? Were there some cases where you were not certain, at once, which was larger? How did you decide? (When it is not obvious which of these objects was larger, you could place one object flat on the other and observe that if they were not the same size, then the smaller would probably fit in the interior of the larger.)
12. Will the original size of a sheet of your notebook paper change if you fold it into four parts? (No) How will you test to see if it has remained the same size? (Place the folded sheet over an unfolded one.)
13. Does the size of your bath towel change when it is wrapped around your body? (No)
14. What happens to the size of a map when you roll it up? (size remains the same)

Length of a Curve or Size of the Surface Enclosed by It

Exploration

Sometimes we compare sizes of objects by comparing lengths and sometimes we compare sizes by comparing the sizes of flat surfaces. Suppose we have pictures of two rectangular fields:





If we wish to compare the amounts of fencing we need to enclose these fields, what property of the rectangles will we compare? (length) Remember that a rectangle is a simple closed curve and if we measure a simple closed curve we are finding its length.

We might, however, be interested in dividing one of the fields so that half would be planted in corn and half in beans. Would we need to know the length of the rectangle? (no) Would it be helpful to know the size of the surface of the field? (yes)

Exercise Set 2

Tell whether you are interested in the length of a simple closed curve or the size of the surface in its interior, or both:

1. To trim the edge of a handkerchief with lace. (length of curve)
2. To buy a rug to cover the living room floor. (both)
3. To buy a desk blotter for your desk. (both)
4. To put a book cover on your text. (both)
5. To string enough beads for a necklace. (length of curve)
6. Can you give 3 other examples of situations in which you would need to know the size of the surface enclosed by a simple closed curve rather than just the length of the curve? (to buy material to make a slipcover for a chair; to fit a table top with a glass cover; to buy linoleum for a kitchen floor, etc.)

Region and Area

Exploration

1. Recall that a simple closed curve by our definition is a path having the following properties:

   a. All of its points lie in a plane.

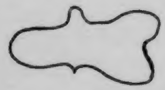
   b. If one traces the path, he eventually returns to the starting point.

   c. The path never intersects itself; i.e., in proceeding once around the path, any point is encountered just once (except for the starting point).

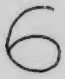
   It also has the property that it separates the plane into three sets of points: the set of points of the curve, the set of points of the interior of the curve, and the set of points of the exterior of the curve.

2. Which ones are simple closed curves? (a, c, g, l)

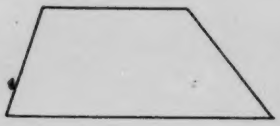
a.



b.



c.



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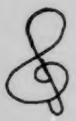
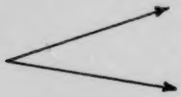


e.



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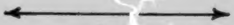
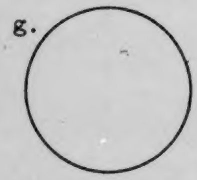
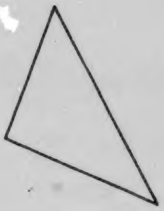


g.



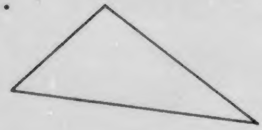
h.



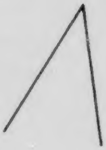
The union of a simple closed curve and its interior is called a plane region. If the curve is a triangle, the plane region is called a triangular region. The union of a polygon and its interior is called a polygonal region. It is the region that we measure when we want to know the size of part of a flat surface.

If the curve is a simple closed curve, trace the curve and color the plane region: (the curves of a, c, d, e, f should be traced and their interiors colored.)

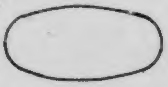
a.



b.



c.



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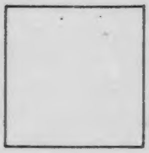
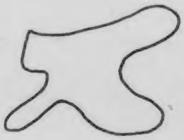
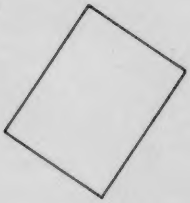


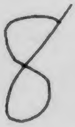
e.



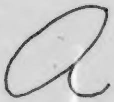
f.



g.



h.



Exercise Set 3

Tell whether you would be interested in area or length, or both, in each of the following:

1. To buy enough wrapping paper for a package. (area - also shape)

2. To decide on the amount of twine needed to wrap a package. (length)

3. To decide on the size of a belt. (length)

4. To buy a piece of land. (area - also shape)

5. To mow a lawn. (area)

6. To run around a closed track. (length)

7. To sail around an island. (length)

8. To tile a basement floor. (area)

9. To measure a triangle. (length)

10. To measure a triangular region. (area)

COMPARING AREAS

Objective: To develop the following understandings and skills:

1. We can compare the areas of two plane regions by seeing whether one region can be fitted into the other region.
2. In making a comparison of two plane regions by inclusion, we may dissect one region and see if the smaller regions can be made to fit in the other region.

Materials:

Teacher: Scissors, large cardboard models of figures to be dissected, paper clips, scotch tape.

Pupils: Scissors, tracing paper, paper clips, scotch tape.

Vocabulary: Dissect

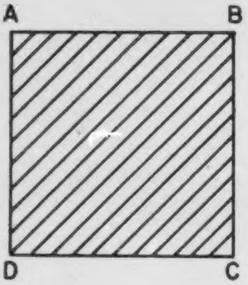
COMPARING AREAS

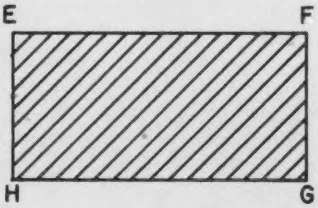
The areas of two plane regions usually can be compared by seeing whether one region may be included in the other. That is, if one plane region can be placed entirely in the interior of the other, then the area of the first region is smaller than the area of the second region.

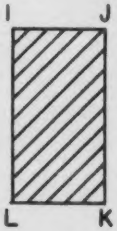
A plane region can be cut up into smaller plane regions. When we cut up a plane region, we say we are dissecting it. To dissect something means to cut it into parts or pieces. Suppose you can dissect one plane region and place all the pieces, without overlapping, entirely on a second plane region. What would this show about the areas of the two regions? (The regions' area is smaller than or possibly equal to the second regions' area.)

Exploration

Which rectangular region has the smallest area? (Region IJKL) Will a tracing of one of them fit into the interior of each of the others? (Yes, region IJKL will.)

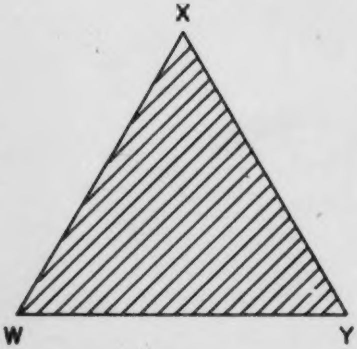


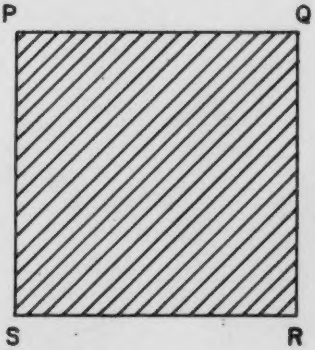




Which rectangular region has the greatest area? (Region ABCD) Will a tracing of either figure fit into the interior of the other? (no) How can the areas be compared? (by seeing whether one plane region may be included in the other.) Cut a tracing of rectangular region EFGH into small pieces. Can all of these small pieces be placed, without overlapping, in the interior of rectangle ABCD? (yes)

Is the area of triangular region WXY less than the area of rectangular region PQRS? (yes)

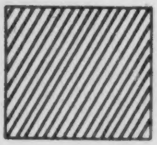




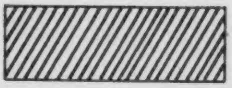
Exercise Set 4

In Exercises 1-3 tell which region of each pair has the greater area. (You may make a paper model of one of these regions and cut it to see if the pieces can be placed, without overlapping, on the other region.)

1.



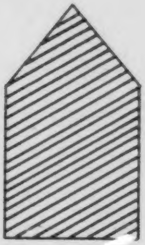
(a)



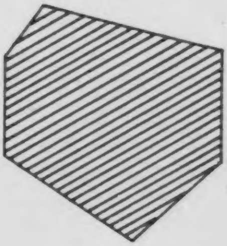
(b)

(a)

2.



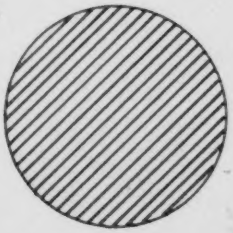
(a)



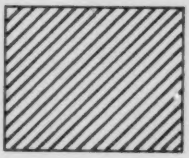
(b)

(b)

3.



(a)



(b)

(a)

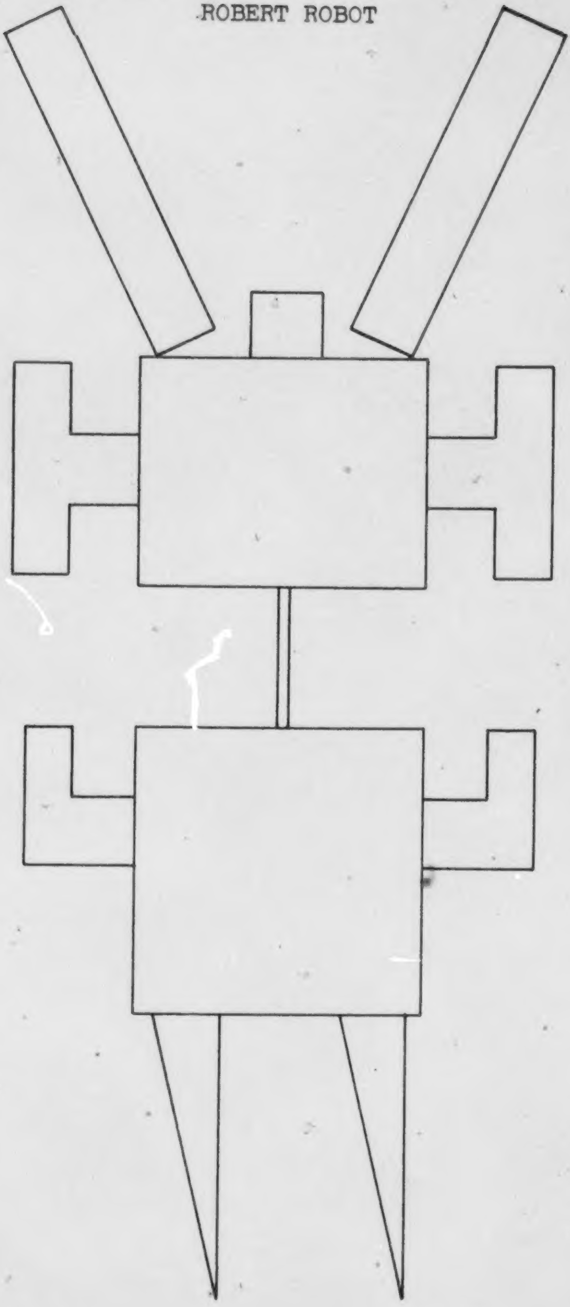
4. Which plane region has the greater area - a region bounded by a square with a side whose length is 3 inches or a region bounded by an equilateral triangle with a side whose length is 4 inches? (square) You will need models of these regions.

P465

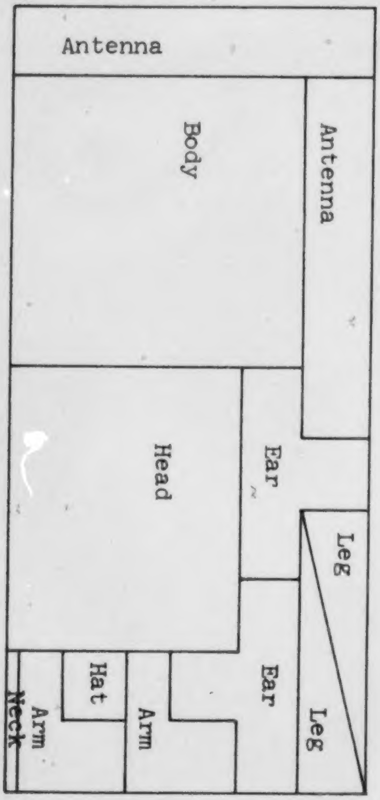
BRAINTWISTER.

Trace "Robert Robot."
Can you arrange the parts of the "robot" in such a way that they form a rectangular region?
The rectangle will have sides whose lengths are $2\frac{1}{8}$ inches and $4\frac{5}{8}$ inches. (answer below)

ROBERT ROBOT



(answer below)


Antenna
Body
Antenna
Head
Ear
Leg
Hat
Arm
Neck
Arm
Ear
Leg

UNITS OF AREA

Objective: To develop the following understandings and skills:

1. In measuring areas a unit plane region is used, as in measuring lengths a unit line segment is used.
2. We need as unit a plane region such that any given plane region can be covered completely by placing these units on it without overlapping.
3. Circular regions (and regions of many other shapes) do not have the covering property needed, but square regions do.
4. As a unit of area we can use a square region whose size is determined by making each side of length 1 unit.

Materials Needed:

Teacher:

1. Large triangular piece of flannel, whose sides have lengths of about 20", 15", and 10". Alternatives: a triangular piece of paper of the same size that can be used on a bulletin board, or a chalkboard drawing.
2. Nine or ten pieces of flannel cut in the form of regions of diameter about four inches. Nine or ten pieces of flannel cut in the form of square regions of side about 4 inches.
3. Straightedge, pieces of string or pieces of wire.

Pupil: For the exploration and for the first part of the next section, each pupil will need about 24 square pieces of construction paper, $1\frac{1}{2}" \times 1\frac{1}{2}"$ in size.

Choosing a Unit of Area

Exploration

> This Exploration does dot appear in the pupils' book. The following questions and outline of procedure may be used as a basis for class discussion. The discussion is summarized in the pupils' book. As you begin the discussion, place the triangular piece of flannel on the flannel board.

This represents a triangular region. What do the edges of the region form? What does each side of the triangle form? Who can tell how to measure the length of a line segment such as the side of a triangle? Will you show the class how you would do this?

> Allow time for child to do this part of demonstration before class. Then draw the following summary from the discussion.
>
> It may be well to add the following for emphasis.

We first choose a line segment as unit. Then we measure this side of the triangle by placing units end to end along it so that they touch but do not overlap. Enough units placed in this way will completely cover the whole side. By counting we find that it takes more than (for example) 4 units but less than 5 units to cover the side exactly. Therefore, we say that the length of this side is greater than 4 units but less than 5 units.

We can measure the area of a plane region such as this triangular region in much the same way. First we choose a plane region to serve as unit.

> At this time ask the children for suggestions how an area of a plane region might be measured by using a plane region as unit. Encourage the children to experiment and demonstrate their ideas at the flannel board. You may wish to give guidance by saying, "Let us think about what sort of plane region would serve best as a unit. We need to cover the whole triangular region by placing enough units next to one another so that they touch but do not overlap."

How would a small circular region do as a unit? Can we cover the whole triangular region by placing these circular regions so that they touch but do not overlap?

Why can we not cover the whole triangular region with circular regions? (Because there will always be little spaces between the circular regions which are not covered.)

Can you think of a plane region that is better than a circular one to use as a unit? (yes, a square region)

> A rectangular region or a triangular region are also acceptable answers, but pupils should be lead to the square region answer in any case.

Why is a square region better than a circular region? (Because we can cover the whole triangular region by placing enough unit square regions side by side so that they touch but do not overlap.)

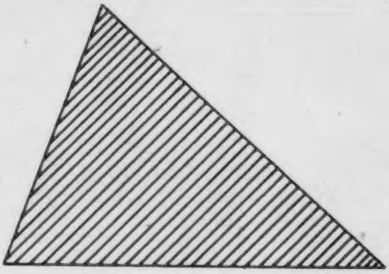
Will someone use these square regions to cover this triangular region?

> Encourage children to illustrate this at the flannel board by using the triangular region and the small square regions.

Let us agree to use a square region as unit of area. Suppose we have already decided on a unit of length. Can we use this unit of length to determine the size of a square region to be used as unit of area? (yes, choose as unit of area a square whose side is 1 unit of length.)

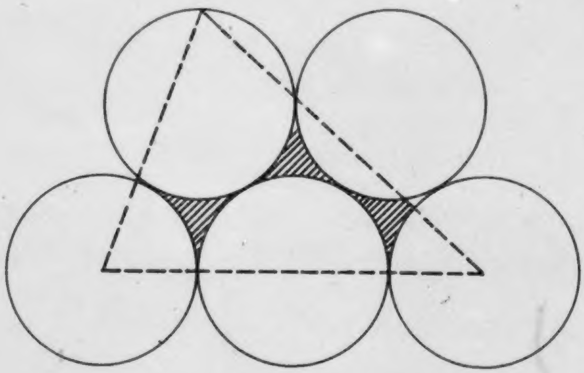
UNITS OF AREA

Choosing a Unit of Area

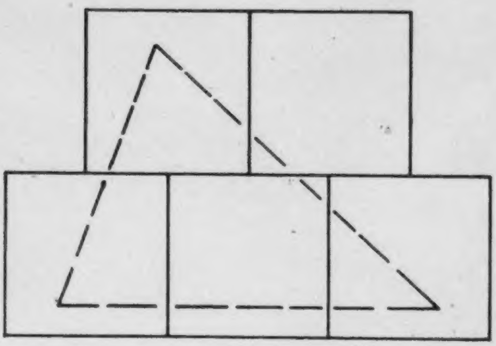


This is a picture of a triangular region. Suppose we wish to measure its area. When we measured the length of a line segment, we needed a unit of length. To measure the area of a region, like this triangular region, we need a unit of area, a unit region.

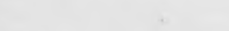
We need to cover the whole region to be measured by placing unit regions on it so that they touch but do not overlap. Is it possible to cover a whole triangular region with circular regions in this way? Why not?



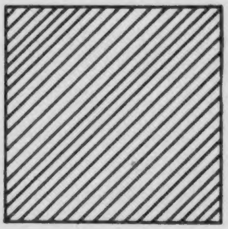
Is it possible to cover the triangular region with square regions? Why



Let us choose a square region as unit of area. We choose a square region whose side is just one unit of length.



unit of length

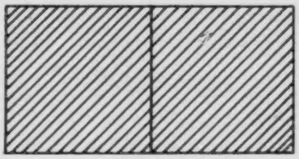


unit of area

Differently Shaped Regions of Same Area

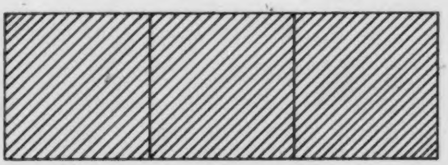
Exploration

Each of you has some square pieces of paper all of the same size. Each piece represents 1 unit of area. Place two pieces side by side on a sheet of paper so that they touch but do not overlap. Trace around the region formed by these pieces. Does your picture look like this?

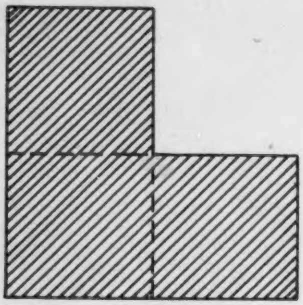


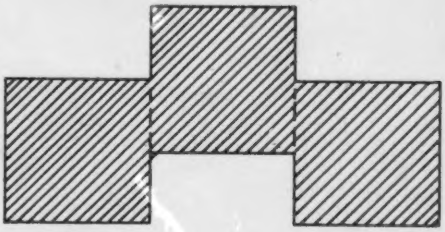
What is the figure you have drawn? (a rectangle) Color the rectangle and its interior. What is the figure you have colored? (a rectangular region) What is the area of this region? (2 units)

Draw and color a rectangular region of area 3 units. Does your picture look like this?

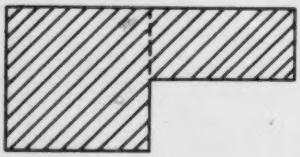


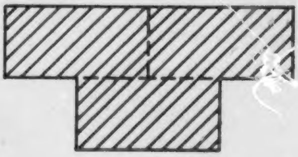
Here are some regions of different shapes, each with area 3 units. Can you think of some others?





Here are some regions of different shapes, each with area of $1\frac{1}{2}$ units. Can you think of some others?





Exercise Set 5

1. Use your square region of paper to trace out and color a region of area 5 units. Make the region any shape you wish.

2. Use your squares of paper (you may want to fold one of them) to trace out and color a region of area $2\frac{1}{2}$ units. Make the region any shape you wish.

3. Take two of your unit square regions of paper and cut each of them into at least three polygonal regions. Now make a new region of different shape, using all your pieces. What is the area of this new region? (2 square regions)

> The teacher may wish to assign additional exercises of this sort.

ESTIMATING AREAS

Understanding and Skills:

1. If a certain number of units can be fitted into a given region without overlapping, then the area of this region is at least this certain number of units.
2. If a certain number of units together cover a given region entirely, then the area of this region is at most this number of units.
3. Just as a scale is useful for measuring lengths, a grid is useful for measuring areas.
4. Smaller units result in more precise estimates of areas.

Materials:

Teacher: Large blackboard drawings of figures from the text.

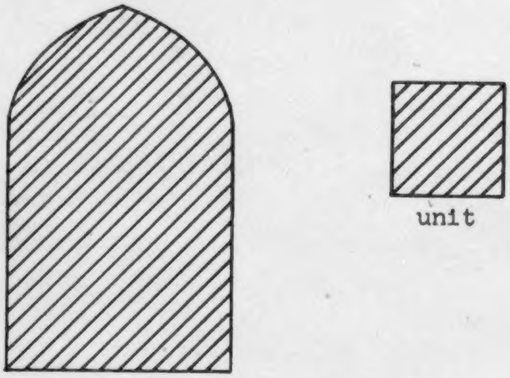
Pupil: The same square pieces of construction paper used in the preceding section, sheets of paper ruled with 1 inch squares, sheets of paper ruled with half-inch squares (that is, squares with sides of length one-half an inch).

ESTIMATING AREAS

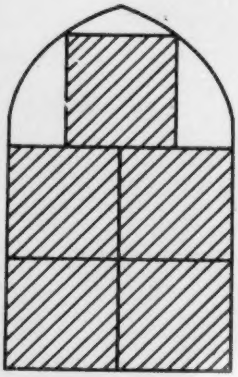
Using Unit Regions to Estimate Areas

Exploration

Suppose we wish to estimate the area of a region with a curved boundary along the top, like a church window, in terms of the unit shown.

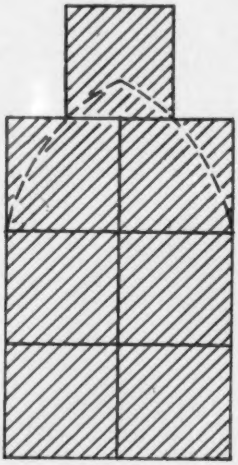


We can fit units into this region as suggested by the picture below.



What does this show about the area of the region? (It is at least 5 units.)

We can also cover this region with unit regions, as shown below.



What does this show about the area of the region? (It is less than 7 units.)

We have not found the exact area of this region, but we now know it is a number of units (not necessarily a whole number) somewhere between 5 and 7.

Can you guess from the picture about what the area is?
(about 6 units; perhaps a little less.)

Exercise Set 6

1. On the next page is a quadrilateral region. See how many of your square pieces of paper you can place entirely on this region. Be sure that no piece goes outside the region and that no piece overlaps another piece. How many pieces are you able to place on the region? (Answers will vary depending upon size of unit chosen)

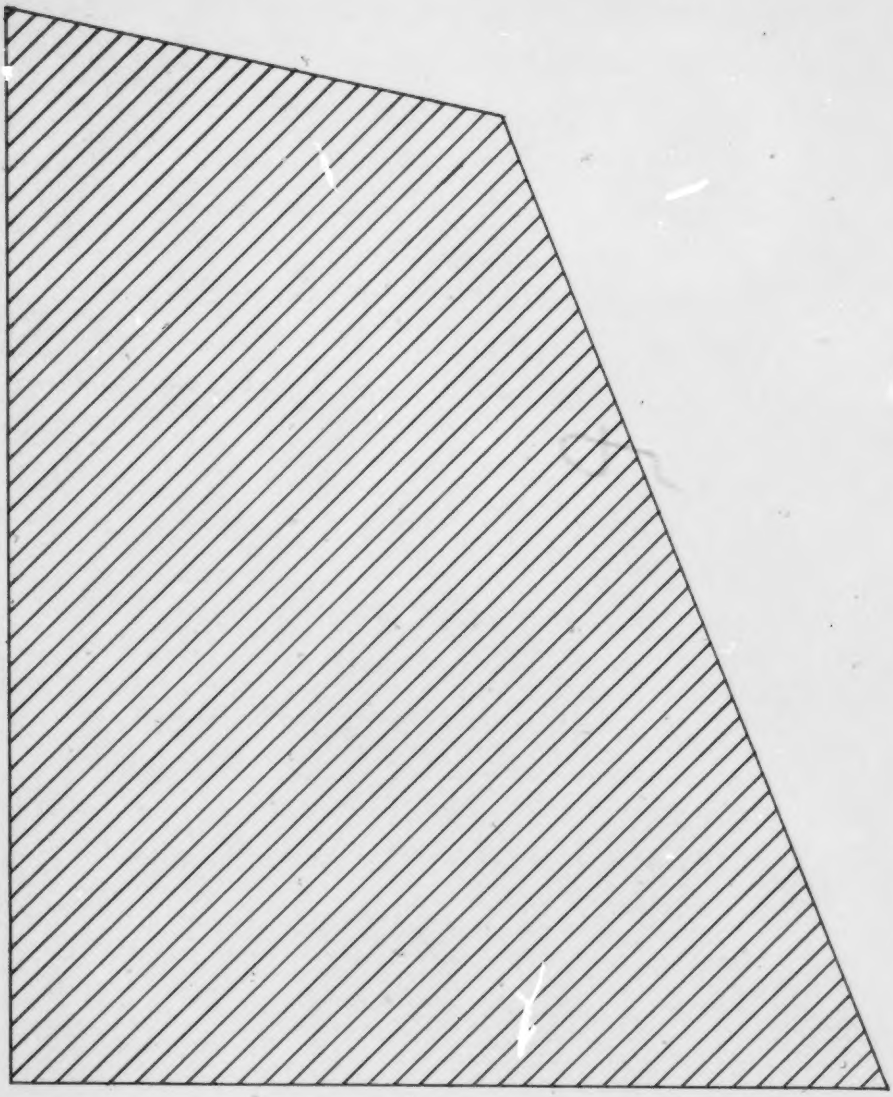
What does this tell you about the area of the region? (It is at least — units; answers might vary)

1. Next, see how many of your square pieces of paper you need to cover the region completely. No piece should overlap another piece. How many pieces do you use to cover the region? (Answers will vary depending upon size of unit chosen.)

What does this tell you about the area of the region? (It is at most — unit.)

Can you estimate about what the area might be? (Answers will vary)

P474



787

3. On the next page is a picture of an oval region. See how many of your square pieces you can place entirely on this region. Be careful that no piece goes outside the region and that no piece overlaps another piece. How many pieces are you able to place on the region?
(Answers will vary depending upon the size of unit chosen.)
What does this tell you about the area of the region? (It is at least — unit; answers will vary.)

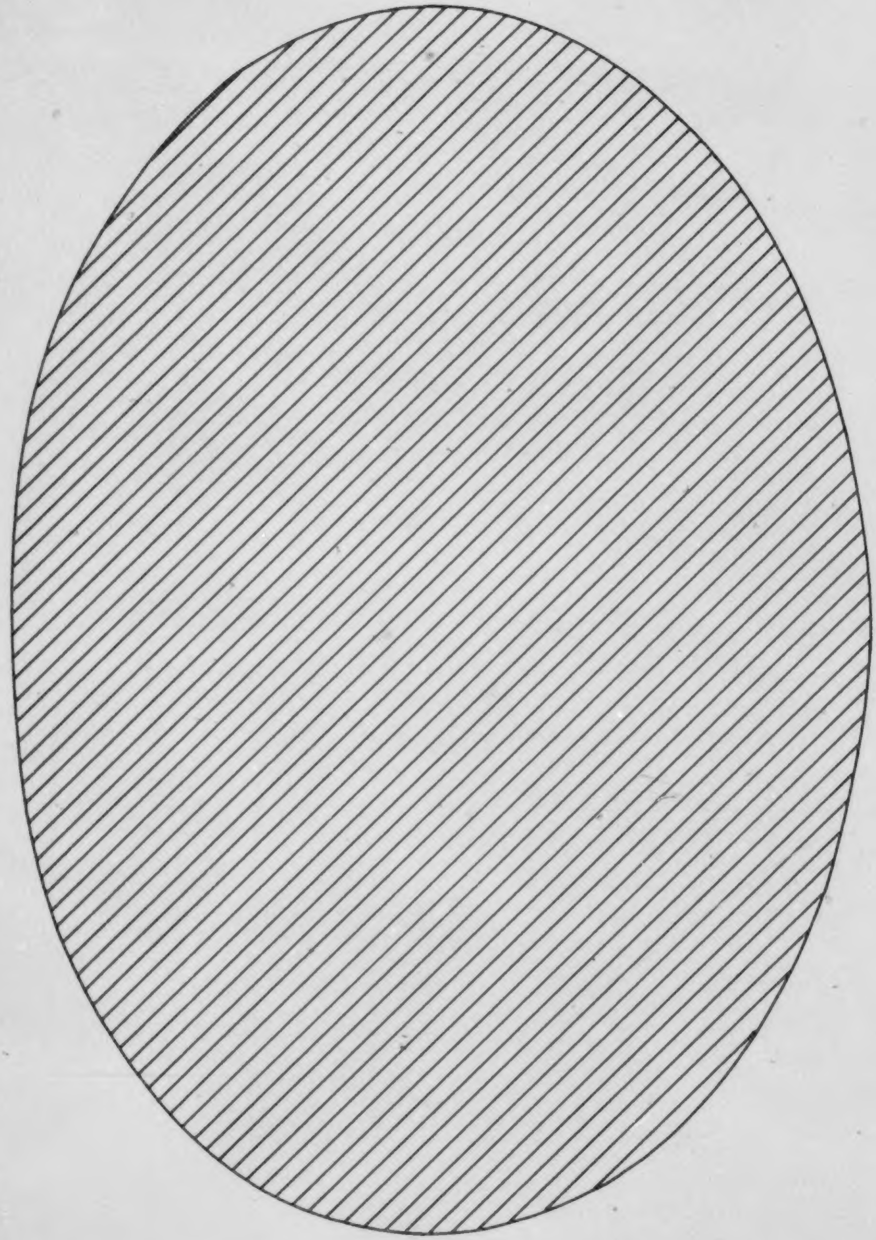
4. How many of your square pieces of paper do you need to cover the region completely? (Answers will vary depending upon the size of unit chosen.)

What does this tell you about the area of this region? (It is at most — unit; answers will vary.)

Can you estimate about what the area might be? (Answers will vary.)

Your class may need more exercises similar to these to establish and maintain skill.

P476



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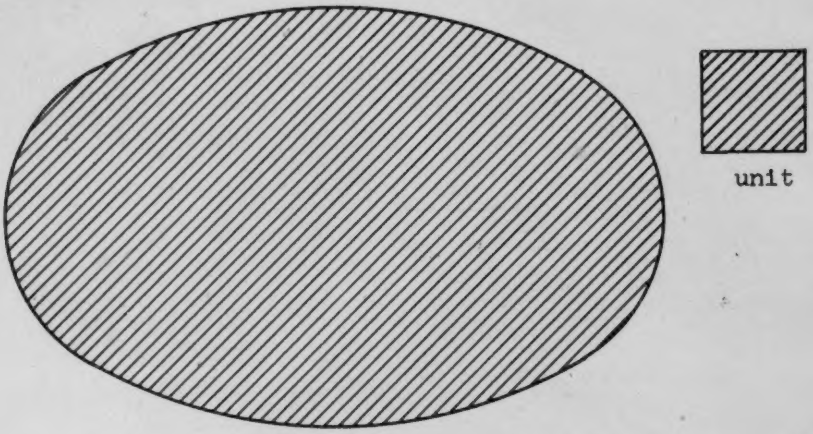
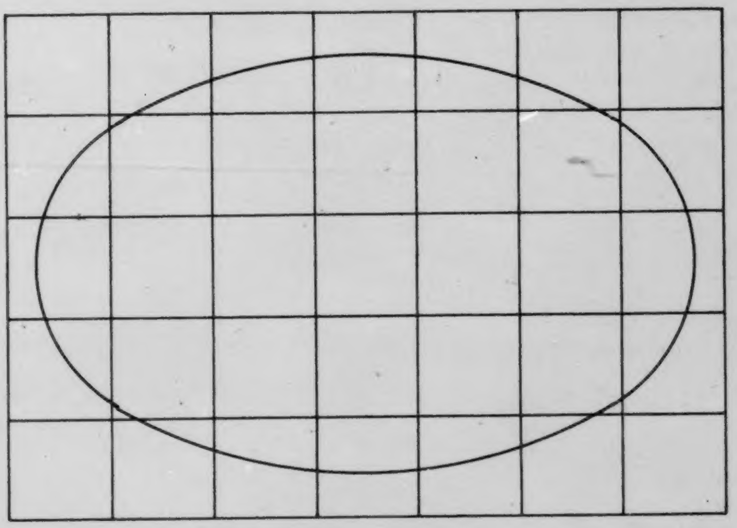
Using Grids to Estimate Areas

Exploration

Suppose we wish to measure the area of the oval region below in terms of the unit shown.

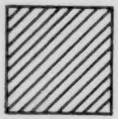


We do not have to use square pieces of paper. Instead we can draw this oval on a grid of units as shown below.

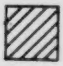


Count the units that are contained entirely in the oval region. How many are there? (11) What does this tell about the area of the region? (It is at least 11 units.) Count the units needed to cover the oval region completely. How many are there? (31) What does this tell about the area of the region? (It is at most 31 units.) The area of the oval region is somewhere between (11) units and (31) units. Looking at the figure, can you guess about what the area would be? (about 18 units; answers will vary.)

We can get a better estimate of this area by using a smaller unit. Suppose we use a new unit of length just half as long as the old one. The resulting old and new units of area look like this:

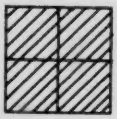


old unit



new unit

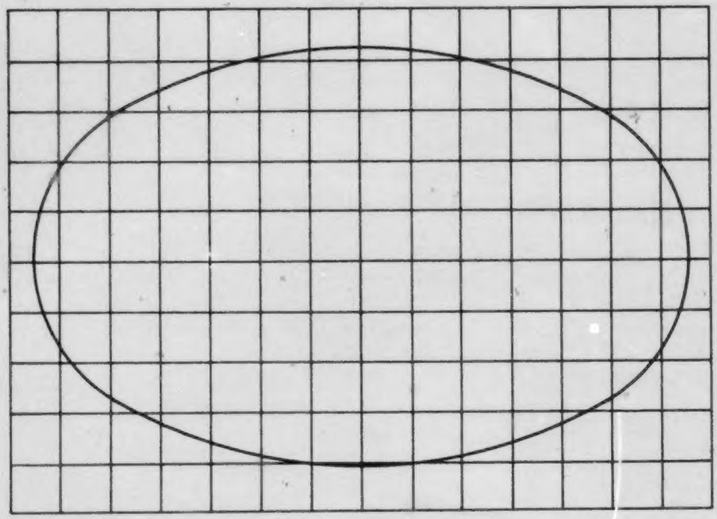
How many new units does each old unit contain? (4)



We have already found that the area of the oval region is somewhere between 11 old units and 31 old units. In terms of the new unit, what does this tell us about the area of the oval region? (It is between 44 new units and 124 new units.)

How do you know? (Each old unit contains 4 new units and 4 × 11 = 44 and 4 × 31 = 124.)

Now let us use a grid of the new smaller units to get a better estimate of this area.



Count the units that are contained entirely in the oval region. How many are there? (69, answers might vary.)

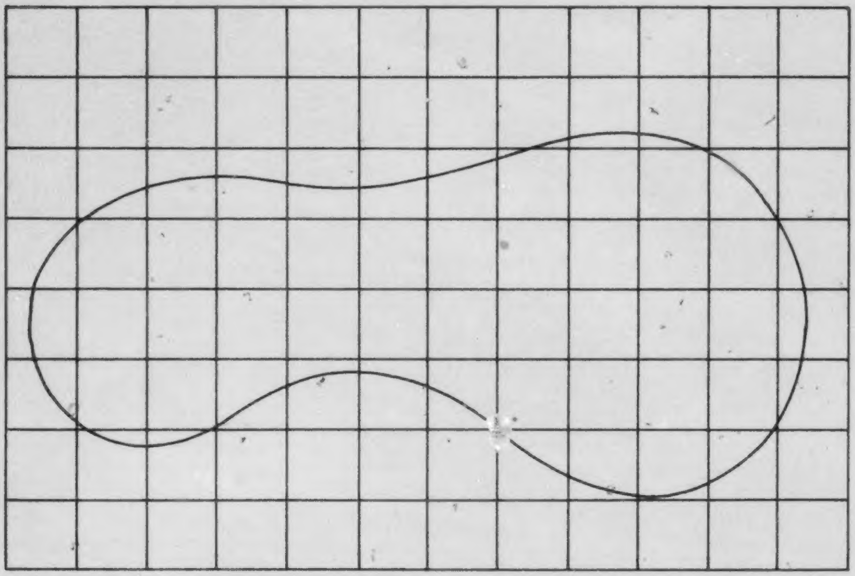
Counting by rows and pointing with the eraser end of his pencil as he counts will help keep the pupil from making errors in counting, and at the same time he will not be writing in his book. It would be an advantage if the pupil could have his own dittoed copy of such figures. Then he could color all the units contained in the oval region, etc. Coloring would help emphasize that area is a property of a region and not of its boundary.

Count the units that are needed to cover the oval region completely. How many are there? (108, answers might vary.) What does this tell about the area of the region? (It is at most 108 units.)

Thus, we now know that the area of the region is somewhere between 69 and 108 new units. Is this better than our old estimate? (yes, the old estimate only told us that the area is somewhere between 48 and 128 new units.)

Exercise Set 7

1. a. Consider the region pictured below on a grid of units.



Fill in the blanks:

There are (23) units contained entirely in the region.

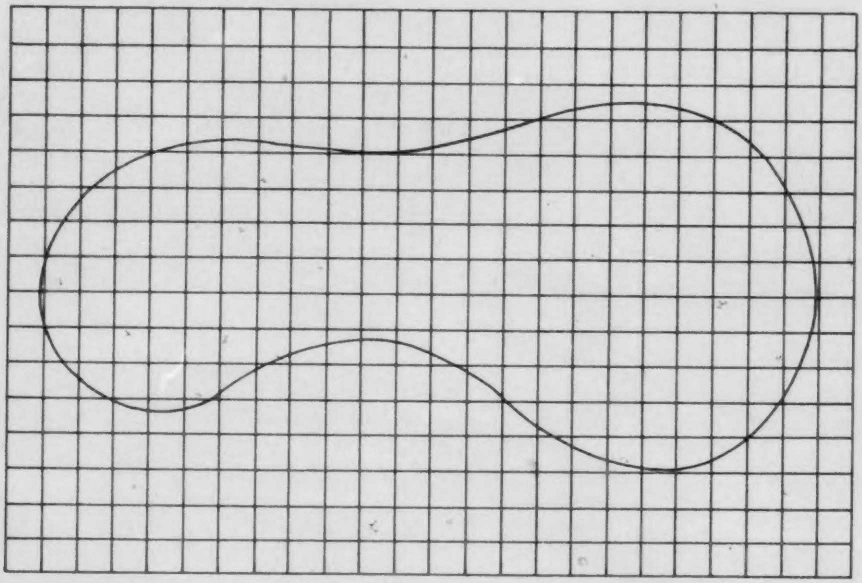
There are (55) units needed to cover the region completely.

The area of the region is at least (23) units and at most (55) units.

Let us choose a new unit of area a square region has as its side a segment just half as long as before. For every old unit of area, we will then have 4 new units of area.

In terms of the new unit, we could say that the area of the region shown on the previous page is at least (92) new units and at most (220) new units. (Since 4×23 = 92 and 4×55 = 220)

b. Consider the same region pictured below on a grid of new units.



Fill in the blanks:

There are (128, answers might vary) units contained entirely in the region.

There are (191 answers might vary) units needed to cover the region completely.

The area of the region is at least (128) units and at most (191) units. (answers might vary)

Is this estimate better than the estimate you made using the larger unit? (yes, the old estimate only told us that the area is somewhere between 108 and 220 new units.)

2. a. On a sheet of paper ruled with 1 inch squares, draw a representation of a simple closed curve. Estimate the area of the region formed by the simple closed curve and its interior.

b. On a sheet of paper ruled with $\frac{1}{2}$ inch squares, trace the simple closed curve you drew in part (a) of this exercise. Estimate the area of the region formed by the simple closed curve and its interior.

c. Which estimate, the one in part (a) or the one in part (b), is the more precise? (The estimate in part (b) is more precise.)

STANDARD UNITS OF AREA

Objectives: To develop the following understandings and skills:

1. One standard unit of area is a square region with 1-inch sides; this unit is called the square inch.
2. Other units obtained similarly are the square centimeter, the square foot, the square yard, and the square mile.
3. An area of 1 square yard is the same as an area of 9 square feet.
4. An area of 1 square foot is the same as an area of 144 square inches.
5. An area of 1 square inch is about the same as an area of 6 or 7 square centimeters.

Materials Needed:

Teacher: Yardstick, colored chalk

Pupil: Sheets of paper (say 8" x 10") marked with grid of 1-inch squares

Before beginning the Exploration, ask pupils to summarize what they learned about standard units of linear measurement, bringing out the following points:

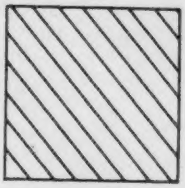
1. Standard units of measurement are needed for convenience and for ease of communication.
2. A unit for measuring the length of a line segment is itself a line segment.
3. The meter is the basic standard unit of length in most countries and in all scientific work.
4. Other standard units of length include the centimeter, the inch, the yard, and the mile.
5. Smaller units permit more accurate measurements.

STANDARD UNITS OF AREA

The Basic British-American Units

Exploration

To measure the area of a region, we first have to choose a unit of area. The most convenient unit of area is a region square in shape. Can you think how hard it would be to talk about areas if each of us chose his own different unit of area? People have found it is simpler if everyone agrees to use the same few units of area. We call these standard units. One standard unit is a square region with sides 1 inch long like this.



We call this unit of area the square inch. Would the square inch be a convenient unit for measuring the area of a sheet of writing paper? (yes)

> At this point the teacher might pass out to each pupil a sheet of paper marked with a grid of square inches (such sheets are commercially available) and ask the pupil to determine by counting what area of the sheet is in square inches. Use sheets whose edges are themselves lines of the grid so that the area is clearly a whole number of grid units.

Would the square inch be a convenient unit for measuring the area of the classroom floor? (No) Why not? (The area of the floor would be such a large number of square inches.)

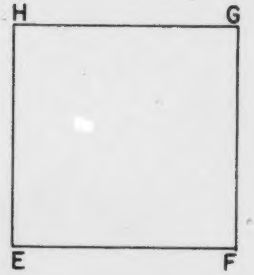
Can you suggest a better unit for measuring the area of the classroom floor? (Yes, the square foot or the square yard.)

Can you explain what a square foot is? (Yes, it is a square region with each side 1 foot long.)

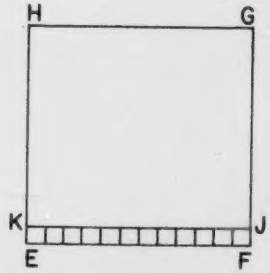
At this point the teacher outlines and and colors a square foot region on the chalkboard. Next pupils are asked if they can explain what a square yard is. After outlining and coloring a square yard region on the chalkboard, the teacher asks pupils to guess the area of a square yard region in square feet. Pupils are then led to suggest drawing a grid of square foot units on the square yard region and counting to determine that a square yard region has an area of 9 square feet. Pupils should also be led to note that a square foot region has an area of $\frac{1}{9}$ square yards.

For the following discussion, the teacher uses full scale models in the form of drawings on paper or on the chalkboard. These should be easily visible to the pupils at their seats.

At the right is a small picture of a square. Your teacher will use a model whose side is actually one foot long. Let us pretend that the length of the side of square EFGH is 1 foot. How many squares of side 1 inch in length could you place, touching but not overlapping, with one side on $\overline{EF}$ as shown in the figure? (12)

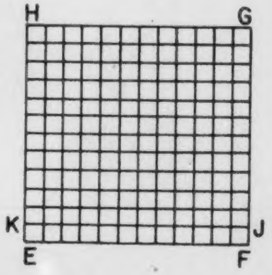


What is the area of region EFJK? (12 square inches.)



How many regions the size of region EFJK could you place in region EFGH? (12)

Since you can place 12 regions the size of EFJK in the region EFGH, and since the area of region EFJK is 12 square inches, then, the measure of region EFGH (where the unit is the region whose area is one square inch) is 12 × 12 or 144.



Thus, an area of 1 square foot is the same as an area of 144 square inches. The area of 2 square feet is the same as an area of 288 square inches. An area of 72 square inches is the same as the area of $\frac{1}{2}$ of a square foot (since $72 = \frac{1}{2} \times 144$).

> A $\frac{1}{2}$ foot square (not $\frac{1}{2}$ square foot) is a square with each side of length $\frac{1}{2}$ a foot. Its area is the same as an area of 36 square inches.

Suppose you wished to measure the area of the whole United States. Would you use the square inch? (no) the square foot? (no) the square yard? (no) Why not? (These units are so small that the area of the United States would be given in inconveniently large numbers)

Can you suggest a better unit for measuring the area of the United States? (yes, the square mile)

> The teacher might ask the pupils to look up the area of the United States as a special assignment.

Exercise Set 8

1. Make a table showing the number of units required to cover these regions:

	Number	Unit
Square, side 1 inch long:	(1)	square inch.
Square, side 1 foot long:	(1)	square foot,
or	(144)	square inches.
Square, side 1 yard long:	(1)	square yard,
or	(9)	square feet,
or	(1296)	square inches.

2. Here are listed areas of some regions. Write each area in at least one other way, using different units.

a. 6 square feet
(864 sq. in.)

b. 4 square yards
(36 sq. ft. or 5184 sq. in.)

c. 10 square feet
1 sq. yd. and 1 sq. ft. or 1440 sq. in.)

d. 288 square inches
(2 sq. ft.)

e. 300 square inches
(2 sq. ft. and 12 sq. in.)

f. 7 square feet
(1008 sq. in.)

g. 32 square feet
4608 sq. in. or 3 sq. yd. and 5 sq. ft.)

h. 1296 square inches
(9 sq. ft. or 1 sq. yd.)

i. 5 square yards
(45 sq. ft. or 6480 sq. in.)

j. 16 square feet
(1 sq. yd. and 7 sq. ft. or 2304 sq. in.)

3. BRAINTWISTER

Find different measures for the area listed below, changing the units of measure as noted:

a. 4 square yards: (36) square feet

b. 5 square feet: (720) square inches

c. 2 square yards 18 square inches: (18) square feet (18) square inches

d. 7 square feet 24 square inches: (1032) square inches

e. 20 square feet: (2) square yards (2) square feet

f. 324 square inches: (2) square feet (36) square inches

g. 2000 square inches: (1) square yards (704) square inches

h. 36 square inches: $\frac{1}{4}$ square foot

i. 2 square feet: $\frac{2}{9}$ square yard

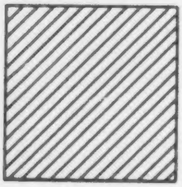
j. 18 square inches: $\frac{1}{8}$ square foot

Area in Square Centimeters

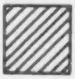
Exploration

All these units--the square inch, the square foot, and the square mile--are units of area in the British-American System of measures. Do you know what system of measures is used in most countries? (yes, the metric system.) What unit of length in the Metric System corresponds most closely to the yard in the British-American System? (the meter) What unit of length in the Metric System corresponds most closely to the inch in the British-American System? (the centimeter) Do you know how the meter and the centimeter compare in size? (the meter is longer than the centimeter; a length of 1 meter is the same as a length of 100 centimeters.) What unit of area in the Metric System would you get by taking a square region with each side 1 centimeter long? (a square centimeter)

Even in Britain and America it is the Metric System that is used for scientific measurements. Therefore, we sometimes need to compare units of area in the British-American System with units of area in the Metric System. Here is a picture of the square inch and the square centimeter.

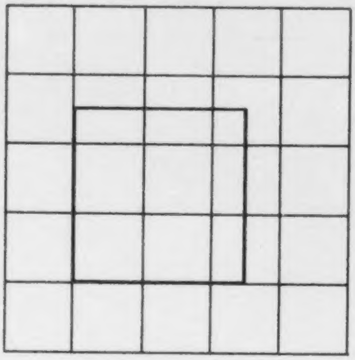


square inch



square centimeter

Which is larger, the square inch or the square centimeter? (the square inch) What would you estimate is the area in square centimeters of the square inch region pictured? (answers will vary) How could you determine this more carefully? (By placing the square inch region on a grid of square centimeter regions.) Here is a square inch shown on a grid of square centimeter regions.



How many of the square regions of the grid are contained entirely in the square inch region? (4)

What does this show about the area of this region? (It is at least 4 square centimeters.)

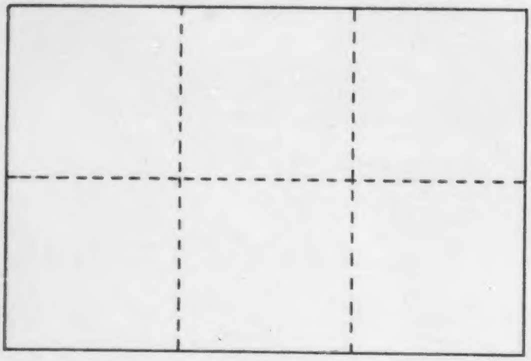
How many of the square regions of the grid are needed to cover the square inch region completely? (9)

What does this show about the area of the region? (It is at most 9 square centimeters.)

Can you now guess this area more accurately? (It is about 6 or 7 square centimeters)

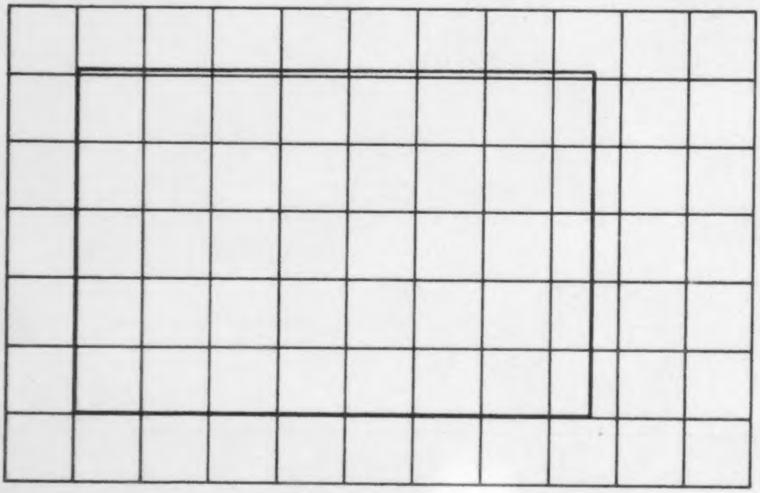
Exercise Set 9

1. Suppose we have a rectangular region with adjacent sides of length 2 inches and 3 inches.



What is the area of the region in square inches? (6 sq inches)

Below is a picture of this same rectangular region on a grid of square centimeter regions. Use this picture to estimate the area of the rectangular region in square centimeters. If you need questions to guide you, look on the next page.



The following questions should help you to find an estimate:

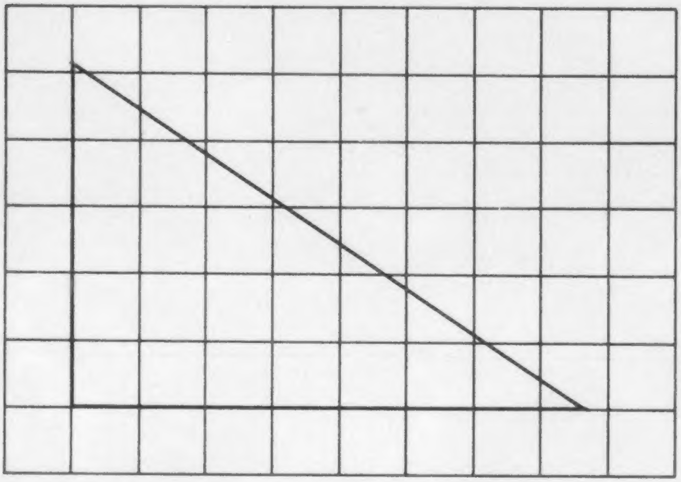
a. How many square regions of the grid are contained entirely in the rectangular region? (35) What does this tell about the area of the region? (It is at least 35 square centimeters.)

b. How many square regions of the grid are needed to cover the rectangular region completely? (48) What does this tell about the area of the region? (It is at most 48 square centimeters.)

c. Can you look at the rectangular region and estimate about what the area would be? (about 39 square centimeters; answers will vary)

d. Fill in the blank: If the area of a rectangular region is 6 square inches, its area is about (39) square centimeters. (answers will vary)

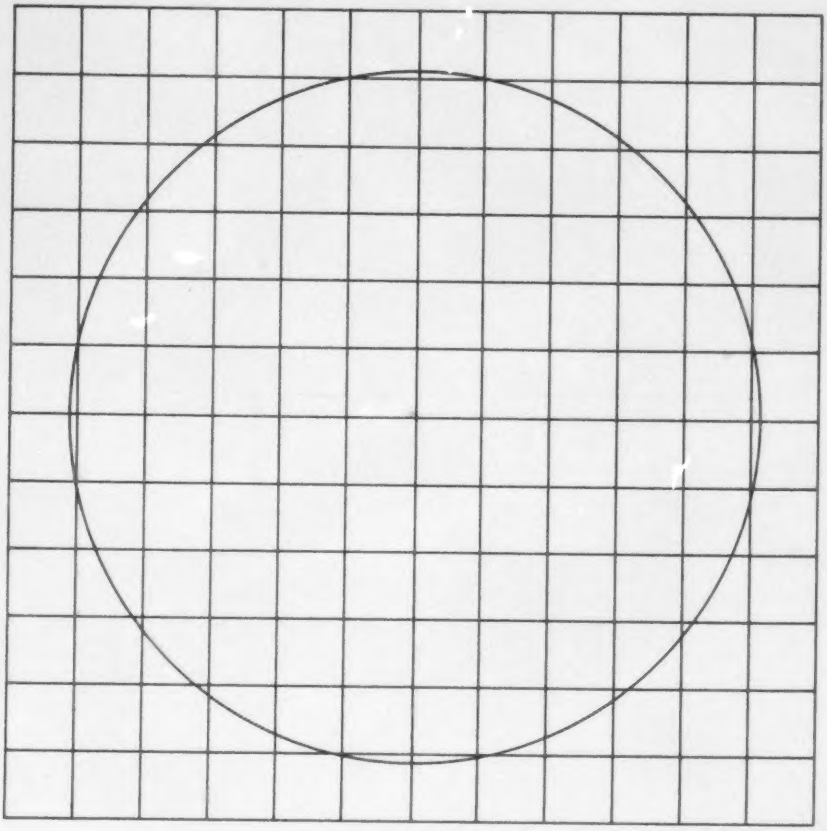
2. Below is pictured a right triangular region on a grid of square centimeter regions. The sides adjacent to the right angle have lengths 2 inches and 3 inches. Find an estimate for the area of the triangular region in square centimeters.



a. The area of the triangular region is at least (14) square centimeters and at most (27) square centimeters.

b. What would you estimate the area would be?
(about 20 square centimeters; answers will vary.)

3. Below is pictured a circular region of radius 2 inches on a grid of square centimeter regions. Find an estimate for the area of the circular region in square centimeters.



a. The area of the circular region is at least (68) square centimeters and at most (112) square centimeters.

b. What would you estimate the area would be? (about 82 square centimeters; answers will vary.)

AREA OF RECTANGULAR REGIONS BY CALCULATION

Objective: To develop the following understandings and skills

1. For each unit of length there is an associated unit which is the square region each of whose sides has length 1 unit.

2. Several differently shaped rectangular regions of the same area can sometimes be formed using a given number of unit square regions.

3. Suppose that in terms of the same unit of length, the measures of the sides of a rectangular region are given whole numbers. Then the product of these numbers is the measure of the region in terms of the associated unit of area. (See note below)

> In simpler terms, item 3 above just says "area is length times width." This simpler formulation is, however, both inexact and incomplete. It is inexact because we don't really multiply lengths and widths; we only multiply numbers. It is incomplete because it does not specify that both length and width must be measured in the same linear unit and that area must be measured in the associated unit of area.

Materials Needed:

Teacher: None

Pupil: Twenty 1-inch squares of construction paper for each pupil, sheets of paper ruled with 1-inch squares.

AREA OF RECTANGULAR REGIONS BY CALCULATION

Building a Rectangular Region

Exploration

You remember that a rectangle has four sides. If we know the measure of any two sides that form a right angle, then we know the measure of all four sides. Why? (because the opposite sides of the rectangle are congruent.) When we speak of "the adjacent sides of a rectangle," we will mean two sides which form part of a right angle.

Earlier in this Unit you found the area of a plane region by covering the region with models of a unit region. What is the shape of a standard unit region? (square)

Your teacher will give you some models of unit regions, each with an area of 1 square inch. Count out twelve of these unit regions. Fit these 12 regions together, without overlapping, so that their boundary is a rectangle. How long are the sides of the rectangle? (There are several possible answers. See the chart below.) See how many different rectangular regions you can form from the 12 square regions and list the information in a chart like the one below:

Lengths of sides		Area
1 in.	12 in.	12 sq. in.
2 in.	6 in.	12 sq. in.
3 in.	4 in.	12 sq. in.
4 in.	3 in.	12 sq. in.
6 in.	2 in.	12 sq. in.
12 in.	1 in.	12 sq. in.

Now use sheets of paper ruled with 1-inch squares. Draw two rectangles of such size that the area of each of the rectangular regions is 20 square inches. (Keep each square inch unit all in one piece.) List the information in a chart as before.

Lengths of sides		Area
1 in.	20 in.	20 sq. in.
2 in.	10 in.	20 sq. in.
4 in.	5 in.	20 sq. in.
5 in.	4 in.	20 sq. in.
10 in.	2 in.	20 sq. in.
20 in.	1 in.	20 sq. in.

What do you notice about the numbers which are the measures of the sides of a rectangle and the measure of its region? (The product of the numbers which are the measures of the sides of the rectangle equals the measure of the rectangular region.)

> In several fourth grade Units, rectangular arrays, as they were called, were used in studying the arithmetic operations on whole numbers. For pupils who have had this approach, the following discussion would be worthwhile.

Where have we already used rectangular arrangements of square regions, quite a while ago? (In first studying operations on whole numbers.) What were these rectangular arrangements of square regions called? (arrays) What were the square regions in an array called? (The elements of the array) How did you learn to calculate the number of elements in an array? (The number of elements in an array is the product of the number of elements in one row and the number of elements in one column.) What are we now calling the number of elements in the whole array? (The measure of the rectangular region) What are we now calling the numbers of elements in a row and in a column of the array? (The measures of two adjacent sides of the rectangular region.) If the number of elements in an array is the product of the number of elements in one row and the number of elements in one column, what does this tell us about the measure of a rectangular region? (It is the product of the measures of the adjacent sides.)

Is the following statement a fair summary of what we have been saying? Two adjacent sides of the rectangle have measures whose product is the measure of the rectangular region. (Yes)

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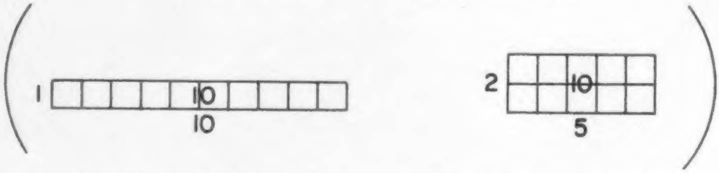
Exercise Set 10

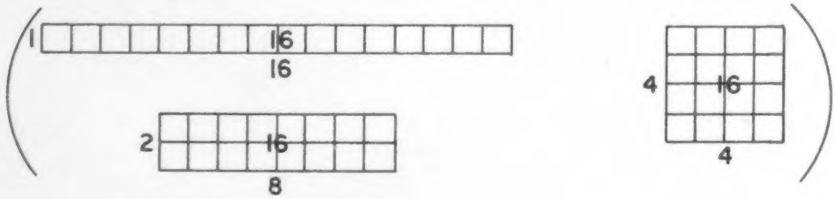
1. Suppose a rectangular region has a measure in square inches of 10. What pair of numbers could be the measures in inches of its sides? (5, 2) Can you think of another pair? (1, 10)

2. Draw on 1-inch squared paper two rectangular regions having a measure of 10 in square inches. On each side write its measure in inches. On the interior of each rectangular region write its measure in square inches.

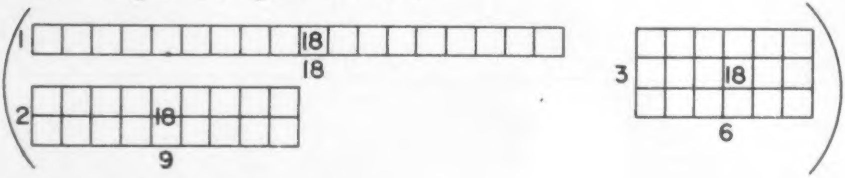


For each of the next three exercises draw rectangles on squared paper and write the measures of the rectangular regions and their sides as in Exercise 2.

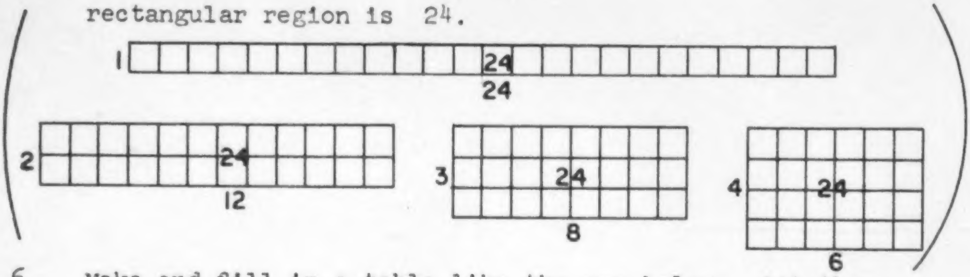
3. Draw three rectangles such that the measure of each rectangular region is 16.



4. Draw three rectangles such that the measure of each rectangular region is 18.



5. Draw four rectangles such that the measure of each rectangular region is 24.



6. Make and fill in a table like the one below. Get the information you need from your drawings in Exercise 3.

Measures in inches of sides of rectangle		Measure in square inches of rectangular region
1	16	16
2	8	16
4	4	16

What do you notice about the product of the measures of the sides in each case? (It is the measure of the region and is the same in each case.)

7. Make and fill in a table similar to that in Exercise 6. Get the information you need from your drawings in Exercise 4.

Measures in inches of sides of rectangles		Measures in square inches of rectangular region
1	18	18
2	9	18
3	6	18

What do you notice about the product of the measure of the sides in each case? (It is the measure of the region and is the same in each case.)

8. Make and fill in a table similar to that in Exercise 6. Get the information you need from your drawings in Exercise 5.

Measures in inches of sides of rectangle		Measure in square inches of rectangular region
1	24	24
2	12	24
3	8	24
4	6	24

What do you notice about the product of the measure of the sides in each case? (It is the measure of the region and is the same in each case.)

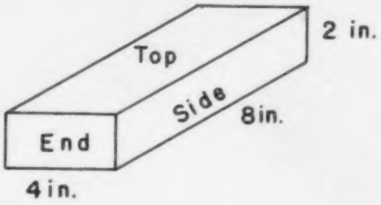
9. Suppose you are given the measures in inches of the sides of a rectangular region. In terms of these given measures, what is the measure in square inches of the rectangular region? (It is the product of these given measures.)

Exercise Set 11

1. Susan made a rectangular doll blanket whose sides were 12 inches and 10 inches long. Find the area of the blanket. (120 sq. in.)

2. Peter made a plywood shelf for his model collection. The shelf was 30 inches long and 8 inches wide. What was its area? (240 sq. in.)

3. Suppose the edges of a brick have the lengths as shown in this picture.



Find these areas:

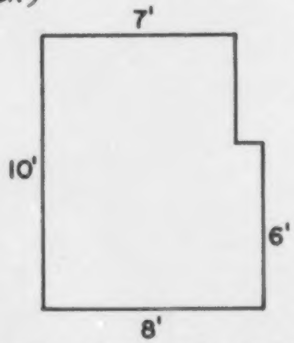
Top of the brick. (32 sq. in.)

Side of the brick. (16 sq. in.)

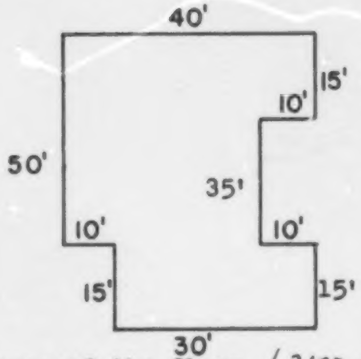
End of the brick. (8 sq. in.)

Total surface of the brick. (112 sq. in.)

4. Here is a picture of a kitchen floor, with the lengths of the edges shown. Find the area of the floor. (Hint: Can you draw a segment which divides the region into two rectangular regions?) (76 sq.ft.)



5. Here is a picture of a floor of a house, with lengths of the edges shown.



a. Find the area of the floor. (2100 sq.ft.)

b. Can you find the area another way? (yes) How? (Ways will vary.)

6. Suppose a high pressure salesman tries to sell you a rectangular lot for your home. After many questions, he reluctantly admits that he has two rectangular lots. One is 3 feet wide and 2000 feet long. The other is 60 feet wide and 100 feet long.

a. What is the area of each lot? (6,000 sq.ft.)

b. Which lot would you prefer? (the one which is 60 ft wide.) Why? (Answers will vary.)

AREA OF A TRIANGULAR REGION

Objective: To develop the following understandings and skills:

1. From every right triangle a rectangle may be formed by properly locating a fourth vertex. The region bounded by the triangle has an area which is one-half that of the region bounded by the rectangle.
2. The measure of a region bounded by a right triangle is found by calculating the product of the measures of the sides of the triangle which determine the right angle, and dividing the product by two.
3. An altitude of a triangle is a segment drawn from a vertex to the opposite side so as to form right angles with this side.
4. The "opposite side" to which the altitude is drawn is called the base of the triangle associated with that altitude.
5. We calculate the measure of a triangular region by taking one-half the product of the measures of an altitude of the triangle and its associated base.

Vocabulary: Altitude, base.

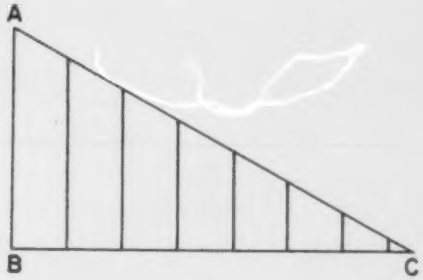
This section challenges the pupil's ability to do some deductive thinking. He needs the skills and understandings of many of the sections in this unit to be able to conclude with a rule for calculating the measure of a triangular region.

AREA OF A TRIANGULAR REGION

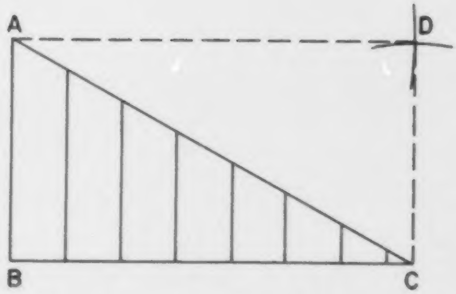
Area of a Region Bounded by a Right Triangle

Exploration

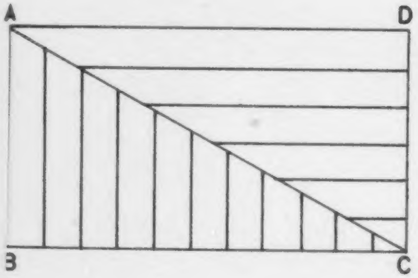
Let us think about how we would find the area of a region bounded by a right triangle.



Using a compass draw a rectangle by making $\overline{AD} \cong \overline{BC}$ and $\overline{CD} \cong \overline{AB}$.

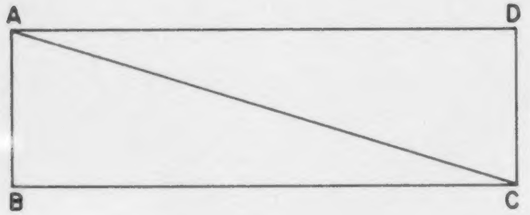


The resulting rectangular region is divided into 2 triangular regions in the following way:



How do the lengths of the opposite sides of the rectangle compare? (opposite sides have the same length) Do you have enough information to be sure that $\triangle ABC \cong \triangle CDA$? (We know that $\triangle ABC \cong \triangle CDA$ since $\overline{AB} \cong \overline{CD}$, $\overline{BC} \cong \overline{DA}$, $\overline{AC} \cong \overline{CA}$ and we thus have three sides of one triangle the same lengths as the corresponding sides of the second triangle.)

If two line segments are congruent, then they have the same length. Similarly, if two triangles are congruent to each other, then the regions associated with them have the same area. Therefore, the area of triangular region ABC is the same as the area of triangular region CDA. The measure of region ABC is what fractional part of the measure of region ABCD? (one-half) What fractional part of the measure of region ABCD is the measure of region CDA? (one-half)



Suppose $\overline{BC}$ has length 10 inches. What is the measure of $\overline{BC}$ in inches? (10) Suppose $\overline{AB}$ has length 3 inches. What is the measure of $\overline{AB}$ in inches? (3) What is the measure, in square inches, of rectangular region ABCD? (3 x 10 or 30) What is the measure, in square inches, of triangular region ABC? (15) Why? (Because the measure of triangular region ABC is one-half of the measure of AB and ½ of 3 = 15) What is the area of triangular region ABC? (15 square inches) What is the measure, in square inches, of triangular region ADC? (15) What is the area of triangular region ADC? (15 square inches)

Summary

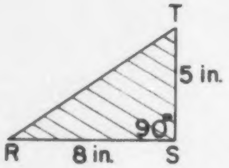
From every right triangle a rectangle may be found by properly locating a fourth vertex. The region bounded by the triangle has an area which is one-half that of the region bounded by the rectangle.

The measure, in square units, of a region bounded by a right triangle is found by calculating the product of the measures, in units, of the sides of the triangle which determine the right angle, and dividing the product by two.

Exercise Set 12

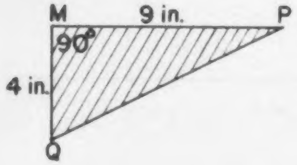
In each exercise the triangle is a right triangle.

1.


T
5 in.
90°
R
8 in.
S

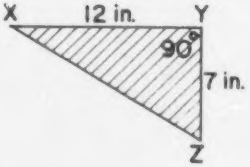
Area of region RST is
(20) square inches

2.


M
9 in.
P
90°
4 in.
Q

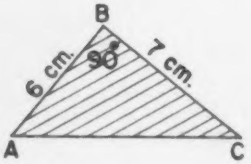
Area of region MPQ is
(18) square inches

3.


X
12 in.
Y
90°
7 in.
Z

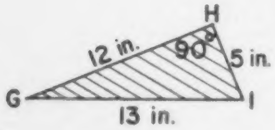
Area of triangular region
XYZ is (42 square inches)

4.


B
6 cm.
90°
7 cm.
A
C

Area of triangular region
ABC is (21 square cm.)

5.


H
12 in.
90°
5 in.
G
I
13 in.

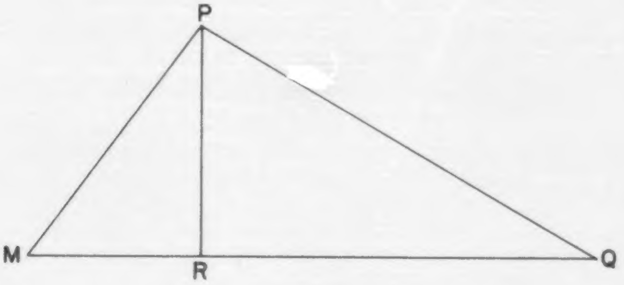
Select the measures you
need and calculate the
area of the region GHI.
(30 sq. in.)

Area of a Region Bounded by a General Triangle

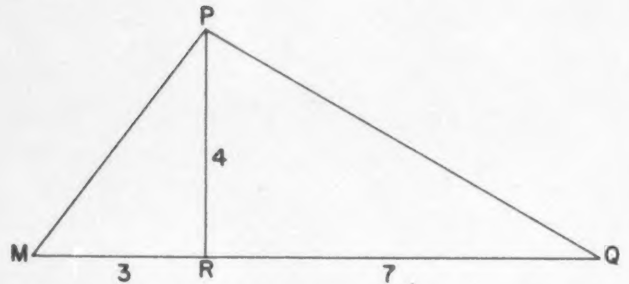
Exploration

Not every triangular region is bounded by a right triangle. We made the area of a region bounded by a right angle depend upon our knowledge of the area of a rectangular region. Now we will make our study of the area of any triangular region depend upon what we have learned about the area of a region bounded by a right triangle. Again we use the concept that area is unchanged when a region is dissected.

If we start with a general triangle such as $\triangle$ MPQ



we may draw $\overline{PR}$ from a vertex P such that $\angle$ PRQ is a right angle and $\angle$ PRM is a right angle. $\overline{PR}$ is referred to as the height or the altitude of $\triangle$ MPQ. The side of the triangle opposite the vertex P is called the base of the triangle. If the altitude is drawn from vertex M, then $\overline{PQ}$ is called the base of $\triangle$ MPQ. If the altitude is drawn from Q, then $\overline{MP}$ is the base. Every triangle has three altitudes and three corresponding bases.



Is $\triangle$ MRP a right triangle? Is $\triangle$ RQP a right triangle?

Suppose the length of $\overline{MR}$ is 3 inches, the length of $\overline{RQ}$ is 7 inches, and the length of $\overline{PR}$ is 4 inches as shown above. What is the area of the region bounded by right triangle MPR? (6 sq. in.) What is the area of the region bounded by right triangle PRQ? (14 sq. in.) What is the area of the region bounded by $\triangle$ MPQ? (20 sq. in.)

Observe:

Measure of region MPR is $\frac{1}{2} \times (4 \times 3)$

Measure of region QPR is $\frac{1}{2} \times (4 \times 7)$

Measure of region MPQ is $\left(\frac{1}{2} \times (4 \times 3)\right) + \left(\frac{1}{2} \times (4 \times 7)\right)$

Using the associative and distributive property:

$$\left(\frac{1}{2} \times (4 \times 3)\right) + \left(\frac{1}{2} \times (4 \times 7)\right) = \left(\left(\frac{1}{2} \times 4\right) \times 3\right) + \left(\left(\frac{1}{2} \times 4\right) \times 7\right) =$$

$$= \left(\frac{1}{2} \times 4\right) \times (3 + 7)$$

$$= \left(\frac{1}{2} \times 4\right) \times 10$$

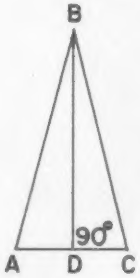
$$= \frac{1}{2} \times (4 \times 10)$$

This tells us that we may calculate the measure of the region MPQ if we take one-half the product of the measure of the altitude and the measure of the base.

Do we get the same measure of the region MPQ if we add the measures of the regions MPR and QPR as we get if we divide the product of the measures of the base and the altitude by 2? (yes)

Exercise Set 13

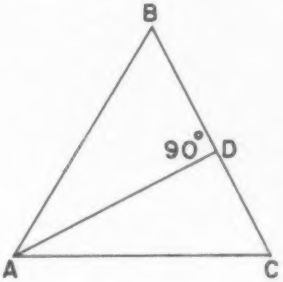
1. $\overline{BD}$ is an altitude. Which line segment is the base? ($\overline{AC}$) Suppose $m\ \overline{BD} = 12$, $m\ \overline{AD} = 3$, $m\ \overline{DC} = 3$, in inches. Find the area of triangular region ABC by two methods.



(1) Measure of region ABD = 18
Measure of region BDC = 18
Measure of region BAC = 36
Area of region BAC = 36 sq.in.

(2) Measure of region BAC = $\frac{1}{2} \times 12 \times 6 = 36$
Area of region BAC = 36 sq.in.

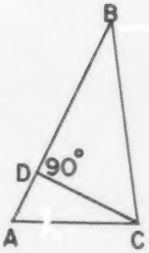
2. $\overline{AD}$ is an altitude. Which line segment is the base? ($\overline{BC}$) Suppose $m\ \overline{AD} = 8$, $m\ \overline{DC} = 4$, $m\ \overline{BD} = 5$, in inches. Find the area of the triangular region by two methods.



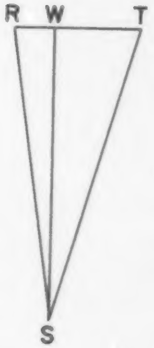
(1) Measure of region ACD = 16
Measure of region ADB = 20
Measure of region ABC = 36
Area of region ABC = 36 sq.in.

(2) Measure of region ABC = $\frac{1}{2} \times 8 \times 9$
$= 36$
Area of region ABC = 36 sq.in.

3. $\overline{CD}$ is an altitude. Which line segment is the base? ($\overline{AB}$) Suppose m $\overline{CD}$ = 6, m $\overline{AB}$ = 12, in inches. Find the area of region ABC. (area of region ABC = 36 sq.in.)



4. $\overline{WS}$ is an altitude. Which line segment is the base? ($\overline{RT}$) Suppose m $\overline{WS}$ = 16, m $\overline{RT}$ = 7, in inches. Find the area of region RST.
(area of region RST = 56 sq.in.)



Chapter 9

RATIO

PURPOSE OF UNIT

The purpose of this unit is to build the understanding that one use of ratio is to indicate how certain physical situations are alike in some respects.

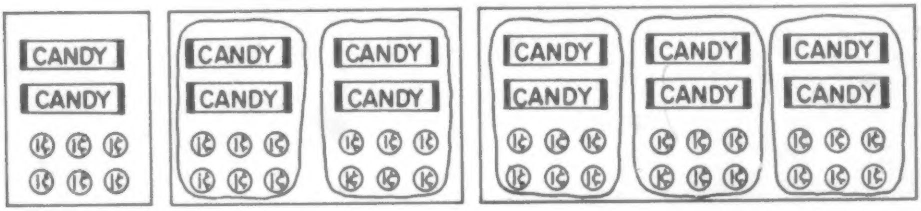
Pupils will learn to extract certain properties from physical situations, and then to express these properties using pairs of numerals.

Children can use the concept of ratio when comparing two sets by noting the correspondence or matching of members of one set with the members of the other set.

MATHEMATICAL BACKGROUND

In other units we have developed mathematics which has proved useful in describing situations in the physical world. The concept of natural number enabled us to indicate one way in which certain sets were alike. We know that a set of 5 apples and that a second set of 5 letters of the alphabet can be put in a one-to-one correspondence. These sets have something in common. We denote the fundamental property in which we are interested by the numeral 5. We have also studied congruence and similarity, concepts which grew out of our desire to compare the size and shape of models of geometric figures. The concept of ratio which we will develop in this unit will give us still one more way of indicating how certain physical situations are alike.

Consider the following problem. I can buy 2 candy bars for 6¢ while you can get 6 of the same candy bars for 20¢. We find ourselves wondering who is getting the better "buy." We shall assume that there is no special discount for large purchases. Since I know that I must present 6¢ for every two candy bars, I can visualize my candy purchasing ability as pictured below. Every 2 candy bars that I buy must correspond to 6 pennies.



The last frame clearly indicates that I am doing better than you are under the given arrangements, for I am getting 6 candy bars for 18 cents but you are paying 20¢ for 6 candy bars.

Exactly how did we reach this conclusion? At first we asked ourselves what sort of purchase would be like the purchase of 2 candy bars for 6¢. The situations represented above are a partial answer. To sharpen our understanding of how these situations are alike, let us summarize the essentials of each situation in a table.

CANDY BARS	2	4	6	8	3
PENNIES	6	12	18		

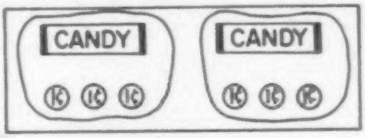
Can we make further entries in our table? If we are able to visualize or draw a picture of the situation, we can make the corresponding table entry with ease.

We notice that an essential aspect of each situation we have described can be represented by using a pair of numerals: (2, 6) for the first frame, (4, 12) for the second, and (6, 18) for the third. These pairs can be used to represent a property common to all of these situations. Using the pair (2, 6) we introduce the symbol 2:6 (read 2 to 6).

> At this point (as suggested by the above table) you may wish to write the symbol $\frac{2}{6}$ instead of 2:6. But the first emphasis is on seeing ratio as a property as another way of comparing two sets of objects and not as a number. Later the correspondence between ratio and rational numbers will be established.

In terms of the above model this can be interpreted as telling us that there are 2 candy bars for every 6 pennies. This same correspondence could have been described using the pair (4, 12) and the associated symbol 4:12. For the above model this would tell us that there are 4 candy bars for every set of 12 pennies. Clearly, 4:12 and 2:6 are different symbols which we can use to indicate the same kind of correspondence and we write 2:6 = 4:12. Once more, as in the case of numerals for fractional numbers, we have an unlimited choice of pairs of numerals to represent the same property. The common property is called a ratio. In the preceding example, the ratio of candy bars to pennies is said to be 2 to 6, or 4 to 12, or 8 to 24.

Can we tell how much I will have to pay for one candy bar? If we take another look at the first frame, as indicated below, we see that one candy bar should cost me 3¢.



Then, 1:3 is another name for this ratio.

How much candy can I buy for 1¢? In trying to answer this question we find ourselves incapable of describing the situation by a suitable pair of numerals unless we consider the candy bars to be divisible. In fact the candy bars are divisible although the store owner is not likely to sell us part of a candy bar. If he would, we would expect to get $\frac{1}{3}$ of a candy bar for a penny. Hence, $\frac{1}{3}:1$ is another name for the ratio we have been studying. However, if the candy store owner won't cut the candy bar into 3 pieces so that each piece is $\frac{1}{3}$ of the bar, this particular pair of numerals doesn't describe a situation that will actually occur at the candy store.

We have seen that 1:3, 2:6, 4:12, 6:18, $\frac{1}{3}:1$ can all be used to describe the basic property that each element of the first set, the set of candy bars, always corresponds to 3 elements of the second set, the set of pennies. We might now ask if we can decide which pairs of numerals can be used to describe this ratio without drawing pictures. Clearly, any pair of the form (n, 3n) where n is a counting number will do. Of course, if the storekeeper will not subdivide the candy bars and if the penny is the smallest unit of money available, only pairs of the form (n, 3n) where n is a counting number will represent actual transactions at the candy counter. That is, 5:15 and $\frac{1}{5}:\frac{3}{5}$ both represent the same ratio. 5:15 tells us that 5 candy bars will cost us 15¢, whereas $\frac{1}{5}:\frac{3}{5}$

tells us that $\frac{1}{5}$ candy bars would cost $\frac{3}{5}$¢. Since the dealer will not sell us $\frac{1}{5}$ candy bar, $\frac{1}{5}:\frac{3}{5}$ does not actually describe a possible exchange of money for candy bars as 5:15 does.

The property described by 2:6 is exhibited in a wide variety of situations and is not restricted to sets of candy bars and pennies. Consider each of the following:

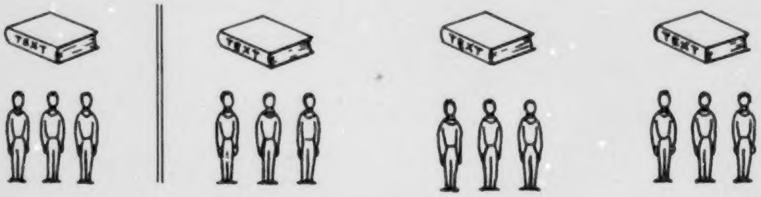
1. There are 2 texts for every 6 students.
2. There are 2 boys for every 6 girls in class.
3. The kart goes 2 miles in 6 minutes.
4. My investment earns \$2 interest for every \$6 invested.

After a brief consideration you will conclude that the table and the associated pictures which we developed for our example of candy bars and pennies would serve equally well to describe each of the above situations. For example, in 1, we have texts instead of candy bars and students instead of pennies.

Consider the statement 1. It describes a situation involving 2 sets: a set of texts and a set of students. The situation in question exhibits a property described by 2:6. We can say that the ratio of number of texts to number of students is 2 to 6. In short, there are 2 texts for every 6 students. Another name for this ratio is 3:9. This indicates that there are 3 texts for every 9 students. 1:3 also describes the ratio of the number of texts to the number of students. However, the ratio of the number of students to the number of texts is 3:1, i.e., 3 members of the set of students correspond to each member of the set of texts. Clearly in making comparisons between numbers of texts and numbers of students it will not be clear that the ratio is 1:3 unless we understand that the first number indicated refers to the set of texts. The order in which the numbers are names is important. Any pair of the form (n, 3n) when interpreted as n:3n could be used to describe the relationship between the set of texts and the set of students. That is, since there are n texts for every 3n students, we have a situation exhibiting the ratio property 1:3. Some pairs of this type are given in the following table. Spaces are provided for further entries.

Texts	1	3	5	12			
Students	3	9	15	36			

We can, of course, never hope to list all possible entries. The pairs indicated in this table are sometimes called rate pairs since they indicate how many texts per student - a distribution rate of texts over the set of students. We can visualize what the table entries tell us about our model sets as shown below.



We might ask how to determine one of our table entries without drawing a picture. In the examples we've been considering the pairs we enter in our table are all of the type (n, 3n) and we see at once that (9, 27) will represent a table entry while (4, 17) will not.

Consider the ratio described by 2:3. This symbol tells us that there are 2 items of the first set for every 3 items of the second. It follows that 4:6, 6:9, $1:\frac{3}{2}$, 100:150, and, in general, 2k:3k would all be other ways of representing this same property. If the first set referred to is the set of boys in school and the second set is the set of girls, we say that there are 2 boys for every 3 girls in school. The symbol 2:3 can also be used to describe a fundamental aspect of what happens when we have a kart which travels at the rate of 2 miles every 3 minutes. The kart travels 2 miles every 3 minutes. In other words, corresponding to every 2 miles stretch covered by the kart, there is a time interval of 3 minutes. The symbol 2:3 will describe the correspondence exhibited here if we choose the elements of the first set to be distances of one mile and the elements of the second to be time intervals of one minute. The aspect of the movement of the kart is equally well described by any symbol of the form 2k:3k. Some such pairs are indicated below. The symbol 40:60 represents "40 miles for each 60 minutes" or "40 miles per hour."

Miles	2	4	5	10	30	$\frac{1}{2}$	1	$\frac{2}{3}$	40
Minutes	3	6	$\frac{15}{2}$	15	45	$\frac{3}{4}$	$\frac{3}{2}$	1	60

Situations in which the correspondence of two sets can be described as above by means of pairs of numerals of the type (ka, kb) or ka:kb all possess a property called the ratio a:b. To each collection of a members of the first set, there corresponds a collection of b members of the second. If two pairs of numerals represent the same ratio, we use an equals sign to show that they are different names for the same ratio. For example,

$$5:10 = 4:8.$$

A statement of this type is called a proportion.

How can we tell if two symbols, for example 6:18 and 8:32, represent the same ratio? The symbol 6:18 tells us that there are 6 members of the first set for every 18 members of the second. This is the same as 1 member of the first set for every 3 members of the second set. That is, 6:18 and 1:3 are different names for the same ratio. Similarly, 8:32 and 1:4 are different names for the same ratio. These symbols 1:3 and 1:4 clearly describe different correspondences and we conclude that 6:18 and 8:32 do not represent the same ratio.

In general, $a:b$ $(a \neq 0)$ and $(b \neq 0)$ represents the same ratio as $1:\frac{b}{a}$ while $c:d$ represents the same ratio as $1:\frac{d}{c}$ $(c \neq 0)$ and $(d \neq 0)$. It follows that $a:b$ and $c:d$ can represent the same ratio if and only if $\frac{b}{a} = \frac{d}{c}$. That is, $a:b = c:d$ if and only if $ad = bc$. Using this test we see immediately that $6:18 \neq 8:32$ for $6 \times 32 \neq 18 \times 8$. Consider a situation in which over a fixed period of time I can earn $1.50 on a $50 investment. From what I know about simple interest, I would expect to get $0.75 on a $25 investment, $0.03 on a $1 investment, etc. If, as before, we use a table to exhibit these results, we would have

Dollars of interest	0.30	1.50	0.75	3	0.03
Dollars invested	10	50	25	100	1

The property common to all of these pairs is the ratio 0.30:10. In particular note the pair (3, 100). This can be interpreted to tell us that we receive $3 of interest for every $100 invested. If we use this pair to describe the ratio property, we write 3:100 and indicate that we get a return of 3 per 100 or 3 per cent. Here cent is used to indicate 100 as it is in the words centennial, century, centipede, etc. We use the symbol 3% (read 3 per cent) to describe how our interest compares dollar for dollar with our investment.

In studying correspondences between two sets, we were led to the concept of ratio. Think about the following statements and you should begin to appreciate the wide applicability of this new idea.

1. The population is 200 people per square mile.
2. The car traveled 100 yards in 6 seconds.
3. The recipe calls for 3 cups of sugar for every cup of water.
4. The scale on this floor plan is $1\frac{1}{2}$ centimeters per 10 feet.
5. I can buy 2 sweaters for $7.
6. My investment is earning 4% interest.

It will be recognized here that ratio is not presented as a number. On the second page of the Mathematical Background ratio is called a property belonging to two sets and the symbol used for ratio describes this property. The similarity of the words "ratio" and "rational" may suggest some close relation between ratios and rational numbers, and such relation does exist. But the full significance of this relationship is not possible until after the study of multiplication and division of rational numbers. The final section of this chapter is intended to establish an awareness in the pupil of some similarity between ratios and rational numbers.

TEACHING THE UNIT

The lessons in this unit are divided in two parts. The first part is of an exploratory nature and is to be developed by the teacher and the pupils working together.

The second part is composed of an exercise set for children to work independently. Each exercise in the set should be discussed with the pupils after the set has been completed. Many of the exercises are designed to carry a bit further the ideas presented in the teacher-pupil exploratory period. These exercises also provide for more clarification of the concepts that are being developed and for practice and drill work.

Each teacher should feel free to adapt the suggestions presented to fit her method of teaching and her group of pupils. As this unit is an introduction to ratio, it is not expected that pupils will develop polished skills, for example, in finding different names for the same ratio. However, many children will develop considerable skill if given an opportunity to do so. The primary purpose is to help all of the pupils develop an understanding of ratio.

INTRODUCTION TO RATIO

Objective: To introduce the meaning of ratio and to show how ratios can be represented by using two numerals.

To develop the understanding that two sets are being compared when we speak of ratio and that it is essential to know what the physical situation is in order to interpret the symbol which describes the ratio.

Materials: Flannel board and cut-outs, a variety of objects which can be placed in two sets and then matched, such as: red and black checkers, sticks, cubes, pencils, scissors, pieces of chalk, erasers, books; in fact, almost any objects commonly found in a classroom

Chalkboard, chalk, pencil and paper for each pupil

Vocabulary: ratio, first number, second number

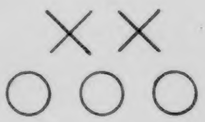
Suggested Teaching Procedure:

Pupil books can be closed for this work. Present a number of physical situations which exhibit the ratio of 2:3. For example, you might start by saying:

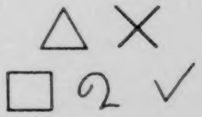
"Here are two sets - a set of chalkboard erasers and a set of pieces of chalk. What number is associated with the set of erasers? Yes, it is 2. What number is associated with the set of chalk? Yes, it is 3. How many sets do we have?" (2)

Follow this same procedure using other physical objects, always having 2 members in the first set and 3 members in the second.

The flannel board could also be used for this exploration. You could also use the chalkboard, drawing such pictures as



or



Any others which the ratio 2:3 describes could also be used.

Your goal is to elicit from children the response that these situations are "alike" in some way. Your principal concern is that the children observe that there are 2 sets and there is a matching in each instance of 2 members of the first set to 3 members of the second set.

Emphasize that the property of the first set having 2 members and the second set having 3 members is called a ratio. The symbol which represents the ratio is written 2:3. Note that 2:3 is a symbol and not the ratio, just as 8 is a numeral and not the number. The symbol 2:3 is read as "two for three" or "two per three" or "two to three."

Show other matchings using the objects, flannelboard, or pictures you draw on the board. Such ratios as 1:3, 5:2, and 3:2 might be used. Children should first identify the two sets being compared, such as "A set of red checkers is the first set and a set of black checkers is the second set" or "A set of pencils is the first set and a set of erasers is the second set." This is to help children be aware of the fact that there are two sets and to be able to identify the two sets. This is necessary because when a ratio is expressed by the symbol, it is necessary to know the set to which each numeral refers.

Children could write the symbols on the board which express the ratio in each case where you are using objects, the flannel board, or pictures on the chalkboard. They should then "read" the symbol, noting especially to what each numeral in the symbol refers. For example, if the two sets are pencils and pieces of chalk and the symbol showing the matching is 1:3, the pupil should say, "The ratio is one per three." (or "one to three" or "one for three") In this case, it means 1 pencil to 3 pieces of chalk.

This ratio is the relationship expressed by "one per three" and it is interpreted in a physical situation as "one pencil per three pieces of chalk" or as "one pencil for each 3 pieces of chalk."

Give children opportunities to read the symbols for ratios such as 8:1 (eight per one, 3:5 (three per five), 7:6 (seven per six). Then ask them to think of a physical situation to go with each of these ratios. A child might say for 8:1, "This is read eight for one and it could stand for eight boys to one girl."

It would be well to go from this symbol of 8:1 to 1:8. Both of these ratios could describe the same physical situation but not in the same way. The first is interpreted as 8 boys to 1 girl while the second is interpreted as 1 girl to 8 boys. In each case, there is a set of 8 boys and a set of 1 girl. Thus it is important that we know the situation from which we have extracted the ratio of 8:1 and 1:8. Exercise 6 in the Working Together section is designed to develop this idea.

After the above development, the pupils might open their books. You could quickly go over this section with them, asking, "Does this tell approximately what we have learned?" In this Working Together or Exploratory section, Examples 1 and 2 focus on the idea of two sets being compared.

Example 3 of the Exploratory section develops the idea of ratio and how the symbol is read.

Examples 4 and 5 give pupils opportunities to study situations that occur in life (physical situation) and to note from them the ratio. Then they write a symbol which expresses this ratio.

In Examples 6 and 7 we develop the idea that it is necessary to know the physical situation. You might now go back to Example 5 and connect, for example, that the ratio symbol for (a) can be either 1:12 or 12:1. But we must know to what the 1 and the 12 refer. This same reasoning applies to Example 8. Thus, all four of these situations can be associated with the symbol 3:5. All that is necessary is that we know to what sets the 3 and the 5 refer.

Exercise Set 1 provides for further clarification of the concepts introduced. For example, Exercise 4 of this set is involved with the idea we've just discussed.

Chapter 9

RATIO

INTRODUCTION TO RATIO

Every day you hear statements like these:

(a) Bill said, "I bought two pieces of candy for four pennies."

(b) "I made two dolls in four days," remarked Mary.

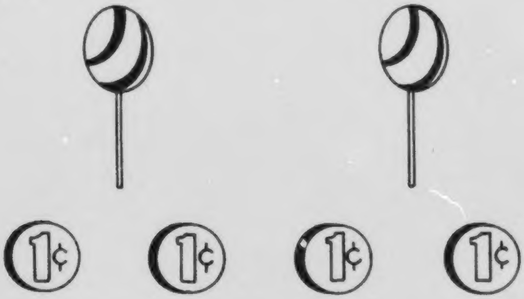
(c) "Jack made two hits in four times at bat," stated Mike.

(d) "My father drove two miles in four minutes," said Helen.

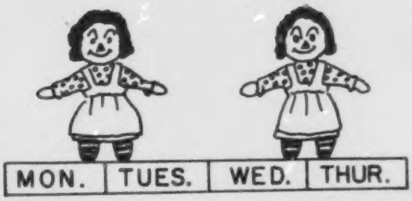
These statements are alike in several ways. Two sets are given in each of them. In the first statement, one of the sets is a set of pieces of candy. The other set is a set of pennies.

1. What are the two sets in statement (b)? (dolls and days) in (c)? (hits and times at bat) in (d)? (miles and minutes)

In each statement, the two sets are matched. In statement (a), 2 candies are matched with 4 pennies. A picture might show it this way:



2. In statement (b), 2 dolls are matched with 4 days.



In statement (c), 2 (hits) are matched with 4 (times at bat).

In statement (d), 2 (miles) are matched with 4 (minutes).

3. In each of the statements 2 members of the first set are matched with 4 members of the second set. This is the idea of 2 to 4 or 2 per 4. Statement (a) matches 2 candies to 4 pennies. Statement (b) matches 2 dolls to 4 days.

In all the statements two things are matched with four things. In statement (c), we say the ratio of the number of hits to the number of times at bat is 2 to 4.

Ratio is a new word to us. It is a symbol which contains two numerals. It is a way of comparing the numbers of two sets of objects.

The way that we express the ratio 2 for 4 is 2:4. This symbol is read "two for four" or "two per four." Two numerals are needed to express a ratio.

Read these ratios:

3:10 4:15 2:5 1:3 5:2 1:2

4. Tom can work two problems in four minutes.

What sets are being compared? (problems and minutes)

What numerals would you write to express this ratio? (2:4)

Jean can work five problems in four minutes.

How would you express this ratio? (5:4)

5. In each of the following, name the two sets that are being compared. Write the symbol for the ratio that compares the two sets.

(a) "I can travel one mile in twelve minutes by using the Boy Scout pace," said Lee. (miles and minutes, 1:12)

(b) The speed limit on the highway is sixty miles per hour. (miles and hours, 60:1)

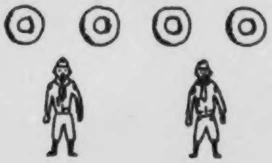
(c) John ate three peaches to Perry's two peaches. (peaches and peaches, 3:2)

(d) Helen won three out of four games. (games won and games played 3:4)

(e) Charles rode his bike to school eighteen times in twenty days. (Times and days, 18:20)

(f) Dick ate lunch at school four of the last five days. (lunch days and school days, 4:5)

6. The cook at the Boy Scout camp said: "I will bake four doughnuts for each two boys."

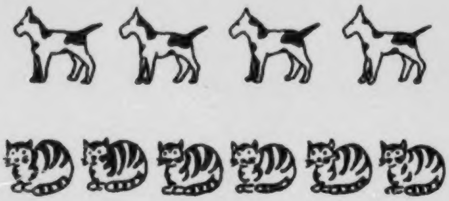


What would you write to express this ratio? (4:2) If the cook said, "I will bake some doughnuts so that there are two boys for every four doughnuts," the ratio would be 2:4. When the cook said "four doughnuts for each two boys" the ratio was 4:2. When he said "two boys for every four doughnuts" the ratio was 2:4.

To understand the symbol 4:2, we need to know that the first number (4) represents the doughnuts and that the second number (2) represents the boys. To interpret 4:2 then, we think "four doughnuts to two boys." This means there will be 4 members of the first set (doughnuts) to 2 members of the second set (boys). The symbol 2:4 means two boys to four doughnuts. It means that there will be 2 members of the first set (boys) for 4 members of the second set (doughnuts).

In order to know what a symbol such as 2:4 could mean, it helps us to know the situation which gives us 2:4. It might be 2 boys to 4 girls, 2 snakes to 4 frogs, 2 ideas to 4 plans. Name some other situations which are 2:4.

6. The picture shown below shows 4 dogs and 6 cats. We can say, "There are four dogs to six cats."



(a) What ratio expresses how the set of dogs compares to the set of cats? (4:6)

We could also say, "There are six cats to four dogs."

(b) What ratio describes the matching of cats to dogs? (6:4)

We can use a pair of numerals in two different ways to describe the same matching. When these are interpreted correctly, they still tell us the same thing - "There are six cats to four dogs." or "There are four dogs to six cats."

8. Which of the following are "3 to 5" matchings? (a, c)
Which of the following are "5 to 3" matchings? (b, d)

(a) There are three bicycles for five children.
(b) For every five boys in Susan's class there are three girls.
(c) Tom has three marbles for every five that Dick has.
(d) The train traveled five miles in three minutes.

Exercise Set 1

1. In what way are these situations alike? (Each shows a ratio 3:1.)

   (a) Henry walks 3 miles an hour.

   (b) In our fifth grade room, we have 3 social studies books for each pupil.

   (c) That big truck can get only 3 miles for each gallon of gasoline.

2. In Exercise 1 name the two sets that are being compared in (a), in (b), and in (c).

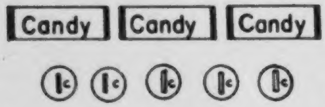
   (a) miles and hours
   (b) social studies books and pupils
   (c) miles and gallons of gasoline

3. Write, in words, how you read each of these symbols:

   (a) 4:1 (four for one or four per one)
   (b) 3:5 (three for five or three per five)
   (c) 1:6 (one for six or one per six)

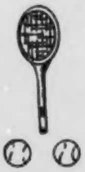
4. Study these pictures. Write a symbol which describes the comparison. Then write a sentence to tell what this symbol means.

(a)


Candy
Candy
Candy
1¢
1¢
1¢
1¢
1¢

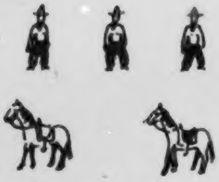
(3:5, You can buy three candy bars for five cents.

(b)



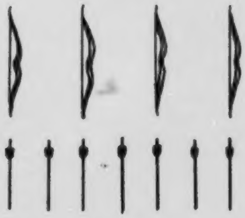
(1:2 There is one tennis racket per two tennis balls.)

(c)



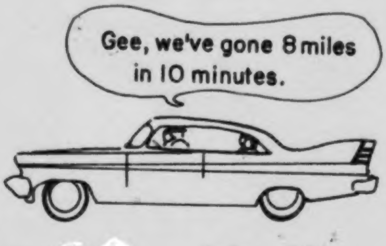
(3:2 There are three cowboys per two horses.)

(d)



(4:7. There are four bows for seven arrows.)

(e)


Gee, we've gone 8 miles in 10 minutes.

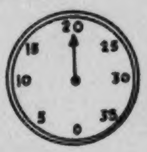
(8:10. We can drive eight miles in ten minutes.)

5. For each of these situations, write a symbol which expresses the ratio.

a) Ned had 2 bee stings for one that Dick had. (2:1)

(b) For our Halloween party, we had 5 sheets of orange paper for 3 sheets of black paper. (5:3)

(c) The speedometer on Steven's bike showed this: (20:1)



(d) Two bags of potato chips cost twenty-five cents. (2:25)

(e) Jean can work four problems in five minutes. (4:5)

6. Draw pictures which could represent comparisons described by these symbols: (The pictures will vary.)

(a) 6:1 (b) 5:2 (c) 2:3

7. Sandra and Mark read this sentence:

On John's farm there are 5 lambs for 3 mother sheep.

Sandra wrote 5:3 to show this comparison. She said, "I know there are 5 lambs for 3 mothers."

Mark wrote 3:5 to show this matching. He said, "I know there are 3 mother sheep for 5 lambs."

Who was correct - Sandra or Mark? (They are both correct.)

DIFFERENT NAMES FOR THE SAME RATIO

Objective: To develop the concept that a ratio has many names and to give practice in finding, in an intuitive way, some of these names.

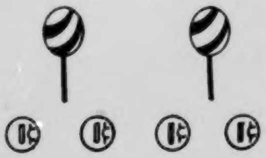
Materials: Flannel board and cut outs, a variety of objects as used in the first section of this unit, chalkboard and chalk, paper and pencil for each pupil.

Suggested Teaching Procedure:

Pupil's books can be closed for this introduction. It might be well to start with the example given in the pupil text. The buying of suckers for pennies has been an experience most children have had.

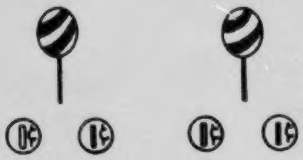
Explain that the two sets are suckers and pennies. Use objects to show this or illustrate on the flannel board.

Arrange the objects like this:



Have a pupil write on the board the symbol that describes this ratio (2:4). Have another pupil read this symbol ("two per four") and still another describe what the symbol means in this problem ("two candies for four pennies"). Ask if another way of matching the suckers to the pennies can be found. Pupils might arrange the sets as four pennies to two candies (which you are really not searching for!) or as one candy to two pennies (which you are searching for!)

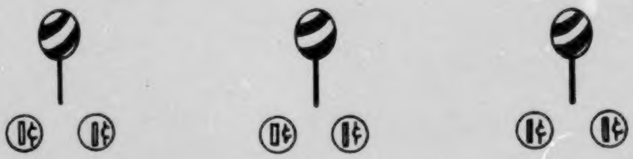
Show the matching as



Lead pupils to see that the matching of two candies to four pennies could be done also as 1 candy to 2 pennies. Have a pupil write on the board the symbol for this ratio. Have another child read it and interpret it in this situation.

Explain that 2:4 and 1:2 are different names for the same ratio.

Show on the flannel board or with objects this matching:



Ask children to write symbols which express the ratio of the number of candies to the number of pennies. They should suggest the symbols

3:6 and perhaps 1:2

Bring out that 2:4 and 1:2 and 3:6 are all names for the same matching. That is, there is always one candy to two pennies.

Use other matchings with different objects to illustrate the idea that a ratio has many names.

Pupil books might then be opened. Teacher and pupils can work together on the exploratory section to be sure the concepts are being developed. Exercise 1 of this section shows several names for the same ratio. Pupils should be able to supply the second number of the symbol on the last part of this exercise. The latter part leads naturally to the idea that there are more different names for a ratio than can be counted.

Exercise 2 of the exploratory section is designed to emphasize the idea of having many names to describe the same ratio. In part (a) there are, of course, many ways of describing the ratio of the number of boys to the number of girls. The name sought here is 10:20. In (b) the name sought is 2:4; in (c) it is 5:10. Part (d) asks for still more names for this same ratio.

Exercise 3 of the exploratory section gives practice in writing symbols for ratios. The symbol for the first ratio might be 8:6 or 6:8, but we must know to which set each numeral refers. There would be many suitable names for this ratio, such as 8:6, 4:3, 16:12, 40:30, and 12:9.

It may be necessary to work with a number of sets of objects or with a number of illustrations with the flannel board or on the chalkboard to develop the idea that there are many names for the same ratio.

Exercise Set 2 gives further opportunity to develop this concept. In each of these exercises, the symbol describing the ratio can be either of two, depending on how the matching of sets is done. For Exercise 1 (a), for example, the symbol can be 1:3 or 3:1 but each of these refer to the same two sets.

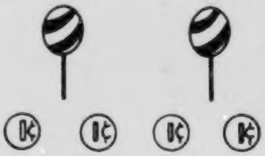
In Exercise 2 of this Exercise Set different names for the same ratio are desired.

Exercises 3 and 4 lead to putting different names for the same ratio in tabular form.

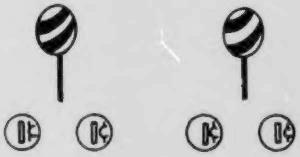
Exercise 7 gives the children an opportunity to be original and creative in their thinking. The basic concept of ratio will be needed here - 2 sets - and a matching of the members of one set with the members of the other. The children's drawings can be three different pictures, each showing the ratio 4:1 or they can use the same sets to show different names for the ratio 4:1.

DIFFERENT NAMES FOR THE SAME RATIO

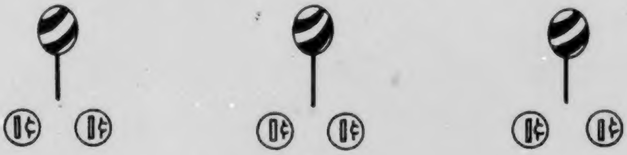
When Bill bought 2 candies for 4 pennies, we described this matching by writing 2:4. We read this, "two to four" or "two per four." We know it means "two candies for four pennies." We drew a picture to represent this.



Can we show this in another way? Look at this picture.

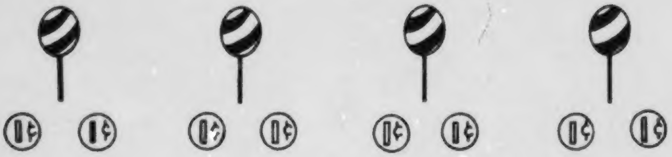


This same matching could be described by the symbol 1:2. This means 1 candy for 2 pennies. Study this picture.



We could also use the symbol 3:6. This means 3 candies for 6 pennies.

Look at this picture.



1. We can describe this in another way by writing 4:8. What does this symbol mean?

The symbols 2:4 and 1:2 and 3:6 and 4:8 are all correct ways of expressing the same comparison. There are many symbols which describe matching. We can refer to it as two per four, or one per two, or three per six, or four per eight. Give other names for this same matching.

Write the second numeral to show other names for the ratio of number of candies to the number of pennies.

5: 10 6: 12

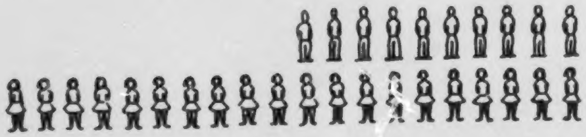
7: 14 8: 16

12: 24 50: 100

How many different names will there be? (More than you can count.)

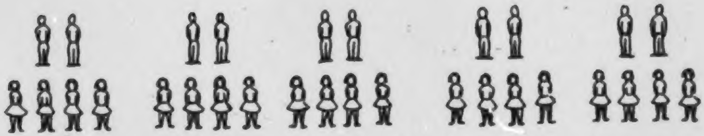
P519

2. Look at this picture of a fifth grade class.



(a) What is one way of writing the symbol which represents the ratio of boys to girls? (10:20)

(b) This picture shows the class lined up in a different way.



Write a symbol to express this ratio of boys to girls. (2:4)

(c) This picture shows still another way of lining up this same class. What symbol expresses this ratio of boys to girls? (5:10)

(d) All of these are names for the same ratio. Complete these symbols to show they are names for the same ratio:

10: 20

2: 4

5: 10

1: 2

50: 100

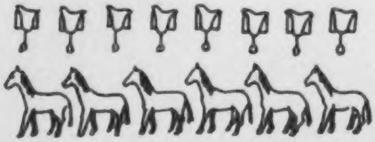
24: 48

214: 428

When we are matching one boy to two girls, there are more names to show this matching than we can count. Whenever we are matching one of a set to two of another set, we usually write 1:2 to express this ratio.

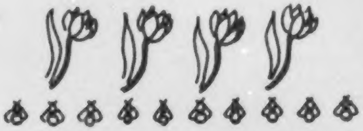
3. For each of these pictures, tell four names for the ratio. (Answers will vary.)

(a) 8 saddles to 6 horses



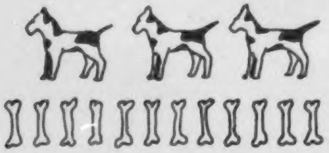
(8:6, 4:3, 12:9, 16:12, etc.)

(b) 4 flowers to 10 bees



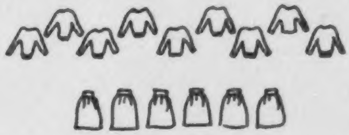
(4:10, 2:5, 6:15, 8:20, etc.)

(c) 3 dogs to 12 bones



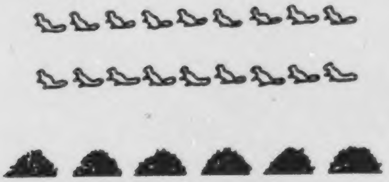
(3:12, 1:4, 2:8, 4:16, etc.)

(d) 9 sweaters to 6 skirts



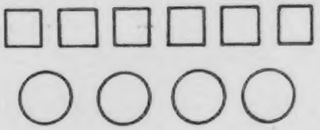
(9:6, 3:2, 12:8, 15:10, etc.)

(e) 18 beavers to 6 beaver houses



(18:6, 9:3, 3:1, 6:2, etc.)

(f) 6 squares to 4 circles

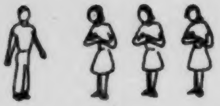


(6:4, 3:2, 9:6, 12:8, etc.)

Exercise Set 2

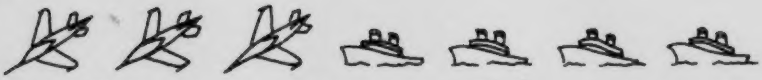
1. You know that a ratio has more names than we can count. Each picture has two sets. Compare the first set to the second set. Write four names for the ratio suggested by the picture. (Answers will vary.)

(a)



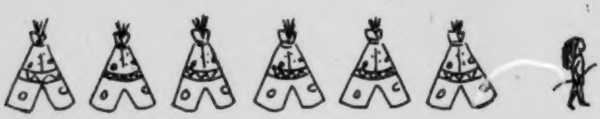
(1:3, 2:6, 3:9, 4:12, etc.)

(b)



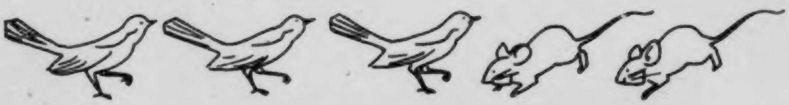
(3:4, 6:8, 9:12, 12:16, etc.)

(c)



(6:1, 12:2, 18:3, 24:4, etc.)

(d)



(3:2, 6:4, 9:6, 12:8, etc.)

2. For each sentence write two names for the ratio suggested by the sentence. Tell what the names mean. (answers will vary.)

   (a) George was going 5 miles per hour on his bicycle.
   (5:1 , 10:2 George is going 10 miles per two hours.)

   (b) In the baseball game, Neil was getting 2 hits for every 5 times at bat. (2:5, 4:10 George was getting 4 hits for every 10 times at bat.)

   (c) The cookies cost 3 for 5¢. (3:5, 6:10 Three cookies for 5¢ or six cookies for 10¢, etc.)

   (d) In Franklin School there are 5 girls for every 4 boys. (5:4 5 girls for every 4 boys,
   10:8 10 girls for every 8 boys, etc.)

   (e) The train was going 4 miles in 3 minutes.
   4:3 4 miles in 3 minutes,
   8:6 8 miles in 6 minutes, etc

   (f) The airplane was going 10 miles in 1 minute.
   (10:1 10 miles in 1 minute,
   20:2 20 miles in 2 minutes, etc.)

3. This table shows several names for the same ratio. Copy it and fill each blank space with the proper numeral.

2:3	
4:6	
6:--	(6:9)
8:--	(8:12)
--:15	(10:15)
12:--	(12:18)
--:21	(14:21)
--:24	(16:24)
18:--	(18:27)
20:--	(20:30)
2n:--	(2n:3n)

BRAINTWISTER

4. This table shows several names for the same ratio. Copy it and complete it.

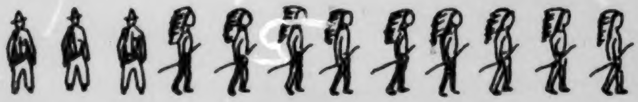
5:4	10:8	15:-- (12)	20:-- (16)	(25) --:20	(30) --:24	35:28	40:-- (32)	(45) --:36

5. Write the letter of each symbol which is another name for the ratio 8:16. (a, b, d, e, f)

(a) 4:8 (c) 1:4 (e) 16:32

(b) 2:4 (d) 3:6 (f) 9:18

6. Draw two pictures of cowboys and Indians like this to illustrate the ratio 3 per 9. (Pictures will vary.)



7. Use any pictures you like. Illustrate with 3 drawings the ratio shown by the symbol 4:1. (Pictures will vary.)

8. What symbol could you write to show the matching of one member from the first set to one member from the second set? (1:1)

9. Express each of these matchings as a number pair using the word "for" or "per."

(a) Jean can work 5 problems in 4 minutes. (5 per 4)

(b) John ate 3 grapes for every 2 that Perry ate. (3 for 2)

(c) The speed limit is 60 miles per hour. (60 per 1)

MORE ABOUT NAMES FOR THE SAME RATIO

Objective: To further develop the child's concept of ratio. To develop some arithmetic way of finding different names for the same ratio

Materials: Flannel board and cut-outs, a variety of objects such as those previously used; 30 slips of blue paper and 30 slips of red paper for each pupil (or any other material, such as colored sticks, so that each pupil might have 2 sets with about 30 members in each)

Suggested Teaching Procedure:

The procedure as presented in the pupil text is in sufficient detail to follow. This can be done with the texts open, teacher and pupils working together.

Comments regarding Examples 1-9, Exploratory section, Pupil Text.

Example 1 is of a review nature.

Example 2 is designed to develop the idea of finding new names for the same ratio. In Exercise 2 (a), (b), (c), because 6:18 and 1:3 are names for the same ratio, we can write

$$6:18 = 1:3.$$

We do not call this a "proportion," but merely indicate that the equal sign tells us that the symbol 6:18 and 1:3 are names for the same ratio. Exercise 3 carries a bit further the idea of different names for the same ratio. Exercise 5 gives the first hint of a way of mathematically determining names for the same ratio. It is here that we will probably need to move slowly so that a good foundation is laid upon which to build in the last part of this unit.

Example 6 has the children working with slips of paper (or other objects) to set up a variety of matching of the two sets. Past experience indicates that children can grasp the concept of ratio quite readily from this type of activity. It would be well to have many exercises based on the matching of these slips of paper. Develop these exercises as is done in the text for Exercise 6.

Example 7 of this exploratory section provides practice in grouping in matching. All children are not expected to be able to solve these without the use of manipulative material. These 10 problems can be solved by using sticks as a "helper."

Examples 8 and 9 give the child an opportunity to demonstrate his understanding of different names for the same ratio.

Exercise Set 3 demands even more abstract thinking from the child. Exercise 3 of Set 3 contains some symbols which cannot easily be determined with sticks or slips or other object.

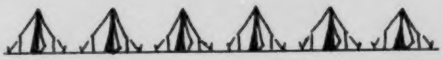
Exercises 4 and 6 of Set 3 give practice in working with mathematical sentences concerned with ratio.

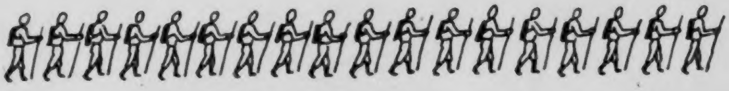
MORE ABOUT NAMES FOR THE SAME RATIO

When we speak of ratio, we immediately think of sets. We know that members of the first set are matched with members of the second set.

1. Name the two sets in each of these situations:
   (a) A fifth grade girl had 8 envelopes for every 12 sheets of paper. (envelopes and sheets of paper)
   (b) The soldiers had 36 bullets for every 2 guns. (bullets and guns)
   (c) The boys rode their bicycles 4 miles in 24 minutes. (miles and minutes)
   (d) At the fifth grade party there were 12 cookies for every 4 children. (cookies and children)
   (e) What symbol names the ratio in (a)? (8:12) in (b)? (36:2) in (c)? (4:24) in (d)? (12:4)

2. Some boys went on a camping trip. There were 6 tents for 18 boys. The same number of boys slept in each tent. Study these pictures.





   (a) What are the two sets? (tents and boys)
   One name for this ratio is 6:18.
   It tells us there are 6 tents per 18 boys.

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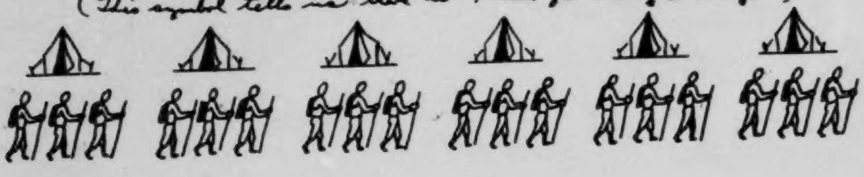
(b) This picture shows us another name for this same ratio is 1:3. What does this symbol tell us?
(This symbol tells me there is 1 tent for every 3 boys.)



Because 6:18 and 1:3 are both names for the same ratio, we can write:

6:18 = 1:3

We can read this mathematical sentence, "Six to eighteen equals one to three" or "Six to eighteen is the same ratio as one to three." In this problem that means "6 tents to 18 boys."

(c) Is this a true mathematical sentence? (yes)

6:18 = 3:9

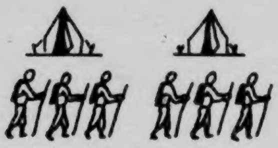
Because 6:18 and 1:3 and 3:9 are all names for the same ratio, we can write

6:18 = 1:3 = 3:9

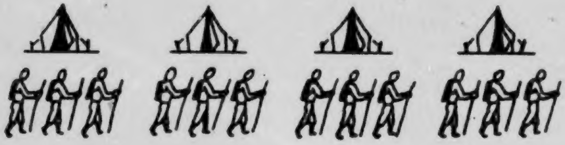
This tells us that 6 to 18 and 1 to 3 and 3 to 9 are names for the same ratio. What is another name for this ratio? (2:6, 4:12, 5:15)

3. These pictures also illustrate this same ratio between numbers of tents and numbers of boys.

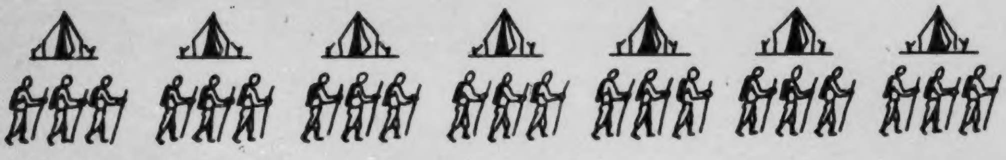
(a) What symbol can we use to express this ratio? (1:3 or 2:6)



(b) Using the picture to help us, what new name can we write for this same ratio? (4:12)



(c) If there were 7 tents, this picture shows how many boys could go camping. What new name expresses this ratio? (7:21)



You see that in every case the ratio is the same.

We still match 1 tent to every 3 boys.

4. Draw a picture to show how many boys could go camping if there were 8 tents of this size. (Answers will vary. There will be eight tents and twenty-four boys. 8:24)

5. Draw a picture to show how many tents would be needed for 27 boys. (The picture will show 9 tents for 27 boys. 9:27)

Here is a table which shows this information:

Tents	1	2	3	(4) --	5	6	(7) --	8	(9) --	(10) --
Boys	3	- (6)	9	12	- (15)	18	21	- (24)	27	30

(a) For this same ratio, how many boys would sleep in 2 tents? (6)

(b) Tell what numbers should be used to fill the spaces in the table. We can use these pairs from the table to write other names for this ratio. For example:

1:3 = 2:6 = 3:9 = 4:12 = 5:15 = 6:18 = 7:21 = 8:24 and so on.

(c) It is not necessary to draw pictures or to make a table to find how many boys could go camping if there were 8 tents. We know that 1 tent will house 3 boys. The members of our sets are tents and boys. We write 1:3.

The symbol 1:3 tells us how the number of tents compares with the number of boys.

We want to find another name for this ratio which has 8 as its first numeral. We write

$$1:3 = 8:N.$$

In this problem we interpret this as "1 tent for 3 boys is the same ratio as 8 tents for how many boys." Instead of 1 tent, we now have 8 tents. Therefore, instead of being able to house only 1 group of 3 boys, we can house 8 groups of 3 boys or 24 boys. $(8 \times 3 = 24)$

The symbol 1:3 and 8:24 are different ways of naming the same ratio. We can write

$$1:3 = 8:24.$$

(d) The symbol 2:6 is another way of describing the ratio of number of tents to the number of boys.

$$2:6 = 8:N.$$

This says that 2 per 6 is the same as 8 per how many. If 2 tents will house 6 boys, then 8 tents, or 4 groups of 2 tents each, should sleep 24 boys. We write

$$2:6 = 8:24.$$

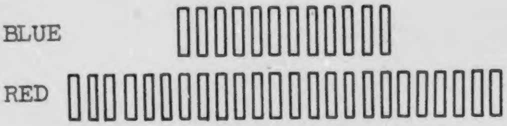
(e) Find the number represented by the letters in the following sentences:

(15) (36) (45)

$1:3 = 5:x$, $3:9 = 12:y$, $1:3 = 15:z$

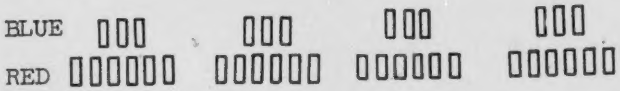
6. Place in two separate piles the 24 slips of paper (red) and the 12 slips of paper (blue) that your teacher has given you. The blue slips are members of one set and the red slips are members of the other set. We will match blue slips to red slips.

(a) Arrange the slips like this:



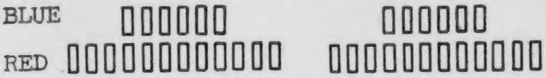
Write a symbol which describes the ratio of blue slips to red slips. (12:24)

(b) Now arrange the slips like this:



Write the symbol which you think best describes the matching of blue slips to red slips. (3:6)

(c) Arrange the slips like this:



Now what symbol would you use to show the matching of blue slips to red slips? (6 : 12)

(d) Arrange the blue slips so that they are in sets of 4. How many red slips would be matched with each set of 4 blue slips? (8) Write the symbol which would best describe this matching. (4 : 8)

(e) Arrange the blue slips so that they are in sets of 2. How many red slips would be matched with each set of blue slips? (4) What symbol best describes this matching? (2:4)

(f) Arrange the blue slips so there is just 1 blue slip to a set. How many red slips would be matched with each 1 blue slip? (2) What symbol expresses the ratio of blue slips to red slips? (1:2)

(g) Complete this table:

Blue slips	12	? (6)	3	? (4)	2	1
Red slips	? (24)	12	? (6)	8	? (4)	? (2)

(h) Replace each question mark so this mathematical sentence is true.

$12:24 = \overset{(6)}{?}:12 = 3:\overset{(6)}{?} = \overset{(4)}{?}:8 = 2:\overset{(4)}{?} = 1:\overset{(2)}{?}$

(i) How many red slips would be matched with 15 blue slips? (30) We know that the matching is 1 blue slip for 2 red slips. So we can write 1:2 = 15:N. We have expressed the idea that 1 blue slip per 2 red slips is the same ratio as 15 blue slips per how many red slips? Instead of one set of 1 blue slip we have 15 sets of 1 slip. Instead of one set of 2 red slips, we have 15 sets of 2 slips or 30 slips. So

$$1:2 = 15:30.$$

7. Find the number represented by n in each of the following sentences. Then write the sentence on your paper. Example: (a) n = 27, 1:3 = 9:27

(a) 1:3 = 9:n
(b) 1:2 = 8:n (n = 16, 1:2 = 8:16)
(c) 1:4 = 5:n (n = 20, 1:4 = 5:20)
(d) 2:5 = 4:n (n = 10, 2:5 = 4:10)
(e) 3:6 = 9:n (n = 18, 3:6 = 9:18)
(f) 3:4 = 15:n (n = 20, 3:4 = 15:20)
(g) 1:4 = 2:n (n = 8, 1:4 = 2:8)
(h) 2:9 = 10:n (n = 45, 2:9 = 10:45)
(i) 4:5 = 24:n (n = 30, 4:5 = 24:30)
(j) 2:1 = 24:n (n = 12, 2:1 = 24:12)

8. Draw a picture to show that this mathematical sentence is true: (Picture will vary.)

1:5 = 3:15

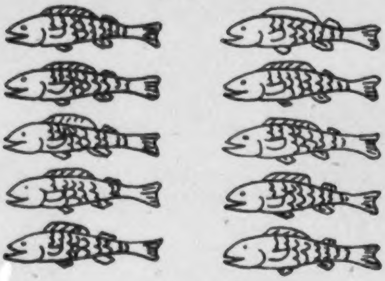
9. Draw a picture to illustrate this:

For every 8 pieces of candy, there were 16 pennies.

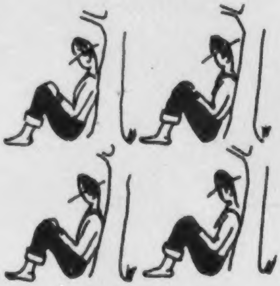
(Picture will vary.)

Exercise Set 3

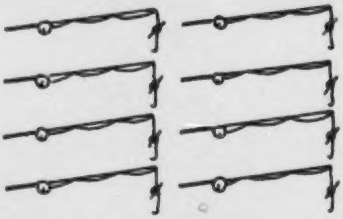
1. (a) Write two symbols which express the ratio of the number of fish to the number of boys. (10:4 , 5:2)



(b) Write two symbols which express the ratio of the number of boys to the number of fish. (4:10 , 2:5)



(c) Write two symbols which describe the ratio of the number of boys to the number of fishpoles. (4:8 , 1:2)



(d) Write two symbols which describe the ratio of fishpoles to boys. (8:4, 2:1)

2. Copy and complete this table.

4:1	16:-- (4)	8:-- (2)	20:-- (5)	36:-- (9)	(100) --:25	12:-- (3)	(24) --:6	(32) --:8	(40) --:10

3. Copy and complete each of these three tables. The last 2 names in each table are braintwisters.

(a)	(b)	(c)
4:8	10:4	6:10
1:--(2)	30:--(12)	(12)--:20
8:--(16)	5:--(2)	(18)--:30
(16)--:32	(20)--:8	30:--(50)
(2)--:4	(50)--:20	24:--(40)
12:--(24)	(60)--:24	(48)--:80
(24)--:48	100:--(40)	(60)--:100
(20)--:40	40:--(16)	54:--(90)
(36)--:72	(80)--:32	36:--(60)
32:--(64)	1,000:--(400)	42:--(70)
3:--(6)	15:--(6)	3:--(5)
(½)--:1	(25)--:10	(9)--:15

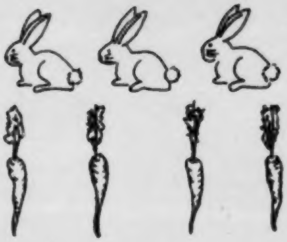
4. Write a mathematical sentence for each of these situations. Then find the answer.

(a) A car will go 20 miles on one gallon of gas. How far will it go on 5 gallons of gas? (20:1 = 5:n
n = 100 miles)

(b) Elmer threw a basketball through the hoop 3 out of 4 times. If he kept this same record, how many times would he need to throw to make 24 baskets?
(3:4 = 24:n , n = 32
He would need to throw the basketball thirty-two times.)

5. (a) Write 4 symbols which express the ratio of the number of rabbits to the number of carrots. (answers will vary. 3:4, 6:8, 9:12, 12:16, etc.)

(b) Write 4 symbols which exhibit the matching of carrots to rabbits. (answers will vary. 4:3, 8:6, 12:9, 16:12, etc.)



6. Write a mathematical sentence for each of these.

(a) 6 per 9 is the same ratio as 2 per 3. (6:9 = 2:3)

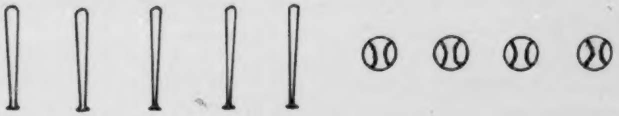
(b) 6 dinosaurs for 4 cavemen is the same ratio as 3 dinosaurs for 2 cavemen. (6:4 = 3:2)

(c) 8 per 3 and 16 per 6 and 24 per 9 are all names for the same ratio. (8:3 = 16:6 = 24:9)

(d) 4 flashlights for 9 boys is the same ratio as 20 flashlights for how many boys? (4:9 = 20:n)

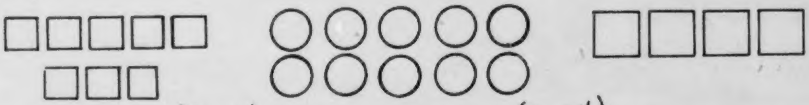
7. Write symbols for ratios suggested by the pictures.

(a)



(5:4)

(b)



(8:10)

(10:4)

(8:4)

USING RATIOS

Objective: To develop an arithmetic method of determining different names for the same ratio

To give pupils an opportunity to use ratio in the solution of problems

Materials: Flannel board and cut-outs, a variety of objects such as those previously used in this unit

Suggested Teaching Procedure:

The presentation in the pupil text is in sufficient detail to be followed.

Examples 1, 2, 3, and 4 of the Working Together or Exploratory section are concerned with the grouping of members of the two sets. If pupils understand this, they will progress rather easily over the latter part, so be certain they comprehend the idea of grouping. The drawing of pictures, encircling the sets, regrouping so the new sets are seen as a separate set, are all helpful devices.

Example 2 presents a method of determining different names for the same ratio without using any manipulative materials. By recalling what they know about factors and primes, children will find this idea a little more understandable. This Example 2 is really the key to the mathematical solution of proportions. It may take many examples to help children understand the idea. Extensive use may need to be made of materials.

Exercise Set 4 is long and contains several Braintwisters. After children have worked independently on the exercises, plan to spend considerable time in working with them in checking their answers. Many of the problems are quite difficult but should serve as a good learning situation. You may want to devise problems of your own to check children's understanding if you work with the pupils on this exercise set as a class activity.

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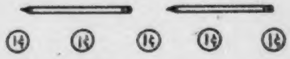
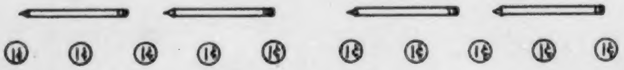
USING RATIOS

1. Alice bought 2 pencils for 5 cents. Write the symbol which expresses the ratio of pencils to cents. If Alice had 10 pennies instead of 5 pennies, how many pencils could she buy? This problem can be solved by drawing a picture such as this:





It can be solved by finding another name for the same ratio, like this:

$2:5 = n:10$

This tells us that 2 pencils per 5 pennies is the same matching as n pencils for 10 pennies.

The symbol n:10 suggests we have 10 pennies and that we want to know the number of pencils to match these. We know that a set of 2 pencils matches a set of 5 pennies and that ten pennies are 2 sets of 5 pennies. So we must have two sets of 2 pencils each to match the 10 pennies. We know, then, that Alice could buy 4 pencils for 10 pennies. You could have solved this problem quite easily "in your head", couldn't you?

2. Could you solve this one "in your head?"

Tom was shooting at a target. He made 5 hits out of 7 shots. If the ratio of the number of hits to the number of shots stays the same, how many hits will he get in 63 shots?

This is a little more difficult to answer. It can be written as:

$$5:7 = n:63$$

This means that five hits per seven shots is the same matching as n hits per 63 shots.

We know we had one set of 7 shots the first time and 9 sets of 7 shots the second time because there were 63 shots the second time. So we have 9 sets of 5 hits or 45 hits per 63 shots.

3. Now let's think about Alice and her pencils. How many could she buy for 25 pennies? You could figure this out "in your head." You also can write a mathematical sentence. The members of the two sets are pencils and pennies. The matching is 2 pencils for 5 pennies. The ratio 2:5 shows how the number of pencils compare with the number of the pennies. We want to know how many pencils Alice can buy for 25 pennies. So we must find another name for the same ratio. We write $2:5 = n:25$. This means 2 pencils per 5 pennies is the same matching as n pencils per 25 pennies. In our second case, instead of 5 pennies

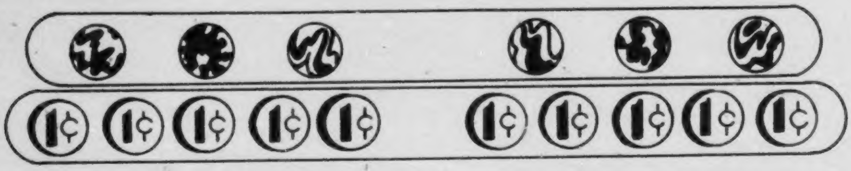
we have 25 pennies or 5 sets of 5 pennies. Therefore, instead of 2 pencils, we will have 5 sets of 2 pencils or 10 pencils. Two different ways of describing the same ratio are 2 per 5 and 10 per 25. The mathematical sentence that says this is $2:5 = 10:25$. Since 10:25 tells us how the number of pencils compare with the number of pennies, we see that we can buy 10 pencils for 25 pennies. To find how many pencils we can buy for 15 pennies, we use the mathematical sentence $2:5 = n:15$. We are asking, "2 per 5 is how many per 15?"

(a) How many sets of 5 are there in 15? (3)

(b) How many sets of 2 pencils should we have? (3) What should n be? (6) If the first numeral refers to pencils and the second to pennies, we see that we should get 6 pencils for 15 cents.

4. Jake bought 6 marbles for 10 cents. How much would 9 marbles cost? Here we have two sets. The members of the first set is marbles and the members of the second set is cents. The ratio of the number of marbles to the number of pennies is 6:10. Our mathematical sentence is $6:10 = 9:n$. In this problem this is interpreted, "Six marbles per 10 pennies is the same ratio as 9 marbles per n pennies." We know that 9 is not a multiple of 6. That is, a set of 9 members cannot be separated into sets of 6 members each.

So let's find some other name for the ratio 6:10. A name which uses smaller numbers might be found. Think of 6 per 10 as shown in this picture.



We see that 6 per 10 is also 2 sets of 3 marbles for 2 sets of 5 pennies. Therefore, 1 set of 3 marbles can be matched with 1 set of 5 pennies.

Exercise Set 4

1. Complete these symbols so that each is a name for the ratio 2:3.

(a) 4:? (6)

(b) ?:12 (8)

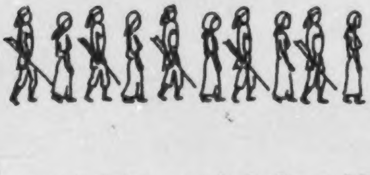
(c) 6:? (9)

(d) 12:? (18)

(e) 100:? (150)

(f) ?:15 (10)

2. This picture is of wagons and pioneers.



How many wagons would be needed for 55 pioneers?

($2:10 = n:55$, $n = 11$
Eleven wagons would be needed for fifty-five pioneers.)

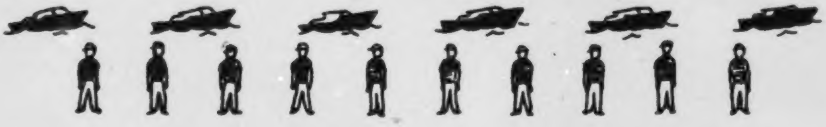
3. Write a mathematical sentence for each of these situations. Let n name the unknown number. Then find the value of n.

   (a) David can run 50 yards in 8 seconds. If he could keep going at this same speed, how long would it take him to run 300 yards? ($50:8 = 300:n$, $n = 48$ It would take him forty-eight seconds to run three hundred yards.)

BRAINTWISTER: How long would it take to run 175 yards? ($50:8 = 175:n$, $n = 28$ It would take 28 seconds.)

   (b) 18 birds live in 10 birdhouses. If the ratio of birds to birdhouses stays the same, how many birds could live in 30 birdhouses? ($18:10 = n:30$, $n=54$ There could be 54 birds living in 30 birdhouses.)

   (c) Study this picture. How many boats would be needed for 30 people? ($6:10 = n:30$, $n = 18$ There would be 18 boats needed for 30 people.)



   How many boats would be needed for 45 people? ($6:10 = n:45$, $n = 27$ There would be 27 boats needed.)

   (d) Glen had 7 ideas in 2 minutes for food for a fifth grade party. If he keeps getting ideas at this same ratio, how many ideas will he have in 8 minutes? ($7:2 = n:8$, $n = 28$ He would have 28 ideas in 8 minutes.)

TRYING SOMETHING NEW

Do you think you could solve some ratio problems like this without using pictures? Using the idea, we know that $6 = 2 \times 3$ and $10 = 2 \times 5$. We see that 6 and 10 have a common factor 2. We can divide both 6 and 10 by 2. When we divide by 2 we get 3 and 5. We can write $6:10 = 3:5$. We know these are names for the same ratio. Let's use this second name for the ratio and write: $3:5 = 9:N$. We are asking, "3 per 5 is the same ratio as 9 per how many?" Now we can see that we have 3 groups of 3 marbles so we need 3 groups of 5 pennies or 15 pennies. Thus, 9 marbles would cost 15 pennies.

1. Look at the mathematical sentence which describes this situation.

   Mr. Smith can drive 50 miles in 1 hour. If he drives at the same speed, how many hours will it take to drive 250 miles? ($50:1 = 250:n$, $n = 5$
   It will take 5 hours.)

2. Try this one on your own.

   Set A is a set of names for the same ratio. Find more names for this same ratio.

   Set A = {18:24, $\overset{9:12}{\underline{--:--}}$, $\overset{6:8}{\underline{--:--}}$, $\overset{3:4}{\underline{--:--}}$}

3. If Chuck eats 3 peanuts for every 2 that Perry eats, how many peanuts will Chuck eat if Perry eats 10?
   ($3:2 = n:10$, $n = 15$ Chuck will eat 15 peanuts)

4. If a car travels 10 miles in 25 minutes, how far will the car travel in 75 minutes? (10:25 = n:75, n = 30 The car will travel 30 miles.)

5. Tell which of these sets are names for the same ratio: (None)

Set A = {20:16, 5:4, 10:8}
Set B = {12:18, 6:9, 2:3}
Set C = {18:24, 3:4, 9:12}
Set D = {32:16, 4:2, 16:8, 2:1, 8:4}
Set E = {48:32, 6:4, 24:16, 12:8, 3:2}

6. If a bank charges 4 dollars for the use of 100 dollars, how much would it charge for the use of 50 dollars? (4:100 = n:50, n = 2 The bank would charge two dollars.)

7. How much would the bank in exercise 6 charge for the use of 250 dollars? (4:100 = n:250, n = 10 The bank would charge ten dollars.)

8. If an airplane flies 570 miles in 1 hour, how far would it fly in $2\frac{1}{2}$ hours? ($2\frac{1}{2}$ hours is 5 half hours, 1 hour is 2 half-hours. 570:2 = 285:1, 285:1 = n:5, n = 1425 The airplane flies 1425 miles in $2\frac{1}{2}$ hours.)

RATIOS AND RATIONAL NUMBERS

You have seen that a ratio such as 2:3 is used to describe a property of two sets. It means that there are 2 objects in one set for 3 objects in another set.

Some other names for the ratio 2:3 are 4:6, 10:15, 20:30, 40:60, and 2000:3000. You could write many more.

In finding other names for the ratio 2:3 you can multiply the numbers 2 and 3 by the same number, if the number is not zero.

The pairs of numerals in 2:3, 4:6, 20:30 represent the same ratio. We can write

$$2:3 = 4:6 \quad \text{and} \quad 2:3 = 20:30 \quad \text{and} \quad 4:6 = 20:30.$$

If we know that $4:5 = n:15$, we can find the number represented by n. It is 12.

Now let us see how some of these things we have just said about ratios are similar to things we can say about rational numbers.

The symbol for the ratio 2 to 3 is 2:3. The symbol for the number two-thirds is $\frac{2}{3}$.

Both symbols use the numerals 2 and 3. Other names for the rational number $\frac{2}{3}$ are

$$\frac{4}{6}, \quad \frac{10}{15}, \quad \frac{20}{30}, \quad \frac{40}{60}, \quad \frac{2000}{3000}.$$

If we know that $\frac{2}{3}$ and $\frac{4}{6}$ and $\frac{20}{30}$ are names for the same rational number, we can write

$$\frac{2}{3} = \frac{4}{6} \quad \text{and} \quad \frac{2}{3} = \frac{20}{30} \quad \text{and} \quad \frac{4}{6} = \frac{20}{30}.$$

In finding other names for the rational number $\frac{2}{3}$ you can multiply 2 and 3 by the same number, if the number is not zero.

If we know that $\frac{4}{5} = \frac{n}{15}$, we can find the number represented by n. It is 12.

You can see that ratios and rational numbers are alike in some ways. After you have studied more about rational numbers, you can see other ways in which they are alike.

Exercise Set 5

1. Write the symbols for two ratios using the numerals 3 and 5. (3:5, 5:3)

2. Write the symbols for two rational numbers using the numerals 3 and 5. ($\frac{3}{5}$, $\frac{5}{3}$)

3. Is 9:10 the name for a number or a ratio? (a ratio)

4. Is $\frac{7}{8}$ the name for a number or a ratio? (a number)

5. Write some other names for 9:10. (Answers will vary. Examples: 18:20, 90:100, 45:50, etc.)

6. Write some other names for $\frac{7}{8}$. (Answers will vary. Examples: $\frac{14}{16}$, $\frac{35}{40}$, $\frac{70}{80}$, etc.)

7. If n:25 is another name for 6:5, what number does n represent? (30)

8. If 3:10 = 18:n, then n represents what number? (60)

9. If $\frac{6}{9}$ is another name for $\frac{n}{81}$, what number does n represent? (54)

10. If $\frac{11}{6} = \frac{110}{n}$, then n represents what number? (60)

Chapter 10

REVIEW

PURPOSE OF UNIT

The purpose of this unit is to provide a review of some of the concepts and techniques which the pupils have learned in Mathematics in Grades Four and Five. It is not necessary to postpone the review provided by this unit until the completion of Chapter 9. Parts of it may be used at appropriate places during the year. For example, the review sections that pertain to the first five chapters of the Fourth Grade might be used after completion of Chapter 5. The optimum method of use can be determined best by the teacher. It is suggested, however, that this entire unit be used as review at the end of Grade 5 although it may have been used "piecemeal" prior to that time.

TEACHING PROCEDURES

The teaching procedure will depend upon the amount of review that is needed. Some pupils may be able to answer all the questions and work all the problems with very little or no assistance from the teacher. Other pupils may have some difficulty. In general, it is suggested that the pupils be given the review on a particular unit without any other preparation than that they had while studying the unit. After the pupils have responded to the review questions in accord with the given directions, the teacher can determine whether there needs to be some instruction to the entire class and to individual pupils. Pupils who have difficulty with items in the review should be encouraged to turn to their copy of the unit which was reviewed in order to correct their own errors. If the pupils have done well enough to indicate no review of a particular unit is needed, they may well undertake the next one.

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Chapter 10

REVIEW

CONCEPT OF SETS

Number the exercises as they are numbered here and write the answers on your paper. If you do not know the answer to an exercise, write the number of the exercise and leave the space beside it blank. Later you may be able to fill in the answers that you did not know.

1. Set A is the set of whole numbers greater than 10 and less than 20. Write the members of A.
(Set $A = \{11, 12, 13, 14, 15, 16, 17, 18, 19\}$)

2. Write a sentence that describes this set:

$$C = \{u, v, w, x, y, z\}$$

(Set C is the set of the last six letters of the alphabet.)

3. If $A = \{3, 6, 9, 12, 15\}$ and $B = \{0, 6, 12, 18\}$ what is $A \cap B$? ($A \cap B = \{6, 12\}$)

4. Using the sets A and B in the preceding exercise, what is $A \cup B$? ($A \cup B = \{0, 3, 6, 9, 12, 15, 18\}$)

5. If R is the set of the states of the U.S.A. that are east of the Mississippi River and S is the set of states that touch the Pacific Ocean, what is $R \cap S$? (the empty set)

NUMERATION

1. Write the letter of each part of this exercise on your paper. Then to the right of the letter write the words, or word, that you would use to fill the blank spaces, or space.

a) 19 = (three) fives and (four) ones.

b) 27 = (five) fives and (two) ones.

c) (four) nines and (four) ones = 40.

d) (five) eights and (no) ones = 40.

e) (five) sevens and (five) ones = 40.

f) (six) sixes and (four) ones = 40.

g) 546 is 5 hundreds, (four) tens, and 6 ones.

h) 546 is 4 hundreds, (13) tens, and 16 ones.

i) 546 is 5 hundreds, 3 tens, and (16) ones.

2. Express 8,471 in 3 different ways as in 1 g), h), i). Letter the three ways a), b), and c).

(Answers will vary.)

3. Write the letter for each part on your paper. If the statement is right, then write yes after the letter. If it is wrong, write no.

a) 3729 is 37 tens plus 29 ones. (no)

b) 734 = 600 + 120 + 24. (no)

c) ten hundreds plus forty tens plus nine ones is the same as one thousand forty-nine. (no; it is 1409)

d) 10,129 = 10 thousands plus ten hundreds plus nine ones. (no)

4. Write the letter for each part on your paper. Then beside it write <, >, or = , whichever makes each a true sentence.

a) 8 + 4 ___>___ 11.

b) (1 + 3) ___<___ (9 + 11).

c) (3 + 4) + 4 ___>___ 1 + (5 + 4).

d) (15 + 14) ___=___ (7 + 5) + 17.

e) (7 + 4) + 2 ___<___ (7 + 5) + 2.

f) (6 + 5) - 2 ___<___ (13 - 7) + 6.

g) (3 × 7) + 9 ___<___ 31.

h) (60 + 3) + (10 - 3) ___=___ (11 - 2).

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PROPERTIES AND TECHNIQUES OF SUBTRACTION, I.

1. Write the letter of each part on your paper. Then beside it write the number represented by n in that part.

   a) $8 + 3 = n.$ ($n = 11$)

   b) $144 - n = 29.$ ($n = 115$)

   c) $n = 1001 - 2.$ ($n = 999$)

   d) $3 - 3 = n.$ ($n = 0$)

   e) $0 + 0 = n.$ ($n = 0$)

   f) $99 + 2 = n.$ ($n = 101$)

2. Write on your paper the letter for each mathematical sentence that is true. (a, d)

   a) $9 + 4 = 13.$

   b) $17 - 9 = 9.$

   c) $88 - 64 = 34.$

   d) $45 + 5 = 50.$

   e) $36 + 37 = 83.$

3. Are some of the mathematical sentences in Exercise 2 false? (yes) If a sentence is false, rewrite it and change one number in it so that it will be true. Letter them the same as in Exercise 2.

   (b) $18 - 9 = 9$ or $17 - 9 = 8$ or $17 - 8 =$
   (c) $88 - 64 = 24$ or $88 - 54 = 34$ or $98 - 64 = 34$
   (e) $36 + 37 = 73$ or $36 + 47 = 83$ or $46 + 37 = 83$

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4. Write the letter of each part on your paper. Then beside it write the number that you would use to fill the blank.

a) If 19 - 10 = 9, then 19 - 9 = (10).

b) If 23 - 11 = 12, then 11 + 12 = (23).

c) If 13 - 10 = 3, then 13 - 9 = (4).

5. Write the letter for each part on your paper. Then beside it write the one of these, > or < , that you would use to make a), b), and c) true sentences.

a) (65 + 42) < (65 + 43).

b) (300 + 700) < (400 + 700).

c) (1300 + 2000) < (1300 + 3000).

6. Write the letter for each part on your paper. Then beside it write the answer to the question.

How many units must be marked on a number line to find z, s, m, p, or n in each of these mathematical sentences?

a) 14 + 17 = z. (z = 31)

b) 139 - s = 40. (s = 99)

c) m = 20 + 40. (m = 60)

d) p + 17 = 30. (p = 13)

e) n - 28 = 13. (n = 41)

7. Write the letter for each part on your paper. Then beside it write the number you would use for the p, q, or r in the mathematical sentence.

a) $p - 8 = 24$ (32)

b) $q = 13 - 4$ (9)

c) $7 - 5 = r$ (2)

d) $20 - p = 12$ (8)

e) $14 - q = 14$ (0)

f) $r = 18 - 18$ (0)

g) $p - 40 = 30$ (70)

h) $p + r = 0$ ($p=0$ and $r=0$)

i) $p + q + r = 0$ ($p=0$, $q=0$, and $r=0$)

j) $10 - p = 10$ (0)

8. On your paper write a mathematical sentence for each problem. Then solve to find n. Write an answer sentence.

a) There were 37 cows in a pasture. Eight of them were black. How many cows were not black?
($37-8=n$ There were twenty-nine cows that were not black.)

b) Jim has 92 coins in a coin folder. It will hold 150 coins. How many more coins will the folder hold? ($92+n=150$ The folder will hold fifty-eight more coins.)

c) Margy practiced her flute lesson for 35 minutes on Monday, 30 minutes on Tuesday, and 45 minutes on Wednesday. How many minutes did she practice on all these days? ($35+30+45=n$ Margy practiced one hundred ten minutes.)

d) A school library had 488 books. The next year 205 books were added. How many books were then in the library? ($488+205=n$ There were six hundred ninety-three books in the library.)

9. Write a different mathematical sentence for each of the following which illustrates how the numbers in each sentence are related.

   a) $7 + 2 = 9$ ($9-2=7$ or $9-7=2$)
   b) $10 - 4 = 6$ ($10-6=4$ or $6+4=10$)
   c) $30 + 30 = 60$ ($60-30=30$)
   d) $x - s = z$ ($z+s=x$ or $x-z=s$)
   e) $n - 5 = 2$ ($2+5=n$ or $n-2=5$)

10. Write on your paper the letter for each sentence that is true. (a, b, c, e)

    a) $20 + 11 = 11 + 20$.
    b) $103 + 301 = 100 + 304$.
    c) $(6 + 5) + 4 = 4 + (6 + 5)$.
    d) $1,207 + 2,011 = 1,102 + 7,021$.
    e) $n + p = p + n$.

11. Some of the statements are true because of the associative property and some because of the commutative property. Write the letter for each part on your paper. Then beside it write associative or commutative to show that you know which property is used.

    a) $2 + (3 + 4) = (2 + 3) + 4$. (associative)
    b) $(18 + 19) + (39 + 12) = (39 + 12) + (18 + 19)$. (commutative)
    c) $(8 + 9) + 6 = (9 + 8) + 6$. (commutative)
    d) $(8 + 9) + 6 = 8 + (9 + 6)$. (associative

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PROPERTIES OF MULTIPLICATION AND DIVISION

1. Copy the letter for each part on your paper. Then beside it write the one of $>$, $<$, $=$ that you would use to fill the blank space so that a) through n) will be true sentences.

a) $5 \times 5 \underline{\ <\ } 4 \times 8$.
b) $6 \times 8 \underline{\ =\ } 8 \times 6$.
c) $9 \times 5 \underline{\ <\ } 6 \times 8$.
d) $94 - 40 \underline{\ =\ } 6 \times 9$.
e) $7 \times 9 \underline{\ <\ } 8 \times 8$.
f) $8 \times 7 \underline{\ <\ } 6 \times 10$.
g) $5 \times 9 \underline{\ >\ } 7 \times 6$.
h) $8 \times n \underline{\ >\ } n \times 7$.
i) $140 - 60 \underline{\ <\ } 9 \times 9$.
j) $9 \times 4 \underline{\ =\ } 6 \times 6$.
k) $8 \times 7 \underline{\ >\ } 9 \times 6$.
l) $p \times 4 \underline{\ =\ } 4 \times p$.
m) $7 \times 7 \underline{>} 6 \times 8 \underline{>} 7 \times 6$.
n) $4 \times 8 \underline{<} 7 \times 5 \underline{<} 6 \times 6$.

2. Copy the letter for each part on your paper. Then beside each letter write the number that you would put in the blank to make a) through h) true sentences.

a) $72 \div 9 = \underline{(8)}$.
b) $32 \div \underline{(8)} = 4$.
c) $\underline{(56)} \div 8 = 7$.
d) $63 \div \underline{(7)} = 9$.
e) $28 \div 7 = \underline{(4)}$.
f) $(8 \times 3) \div \underline{(3)} = 8$.
g) $(12 \div 3) \times 3 = \underline{(12)}$.
h) $(9 \times \underline{(4)}) \div 4 = 9$.

3. We want you to use the Distributive Property of Multiplication. Study this example to see how we rename 17, then how we use the Distributive Property of Multiplication.

$$4 \times 17 = 4 \times (10 + 7) = (4 \times 10) + (4 \times 7) = 40 + 28 = 68.$$

Now write each part on your paper and use the method shown in the example. Part d) is begun for you but it is not finished.

a) $7 \times 12 = 4 \times (10+2) = (4 \times 10) + (4 \times 2) = 40 + 8 = 48$

b) $6 \times 19 = 6 \times (10+9) = (6 \times 10) + (6 \times 9) = 60 + 54 = 114$

c) $7 \times 26 = 7 \times (20+6) = (7 \times 20) + (7 \times 6) = 140 + 42 = 182$

d) $4 \times 153 = 4 \times (100 + 50 + 3) = (4 \times 100) + (4 \times 50) + (4 \times 3)$
$= 400 + 200 + 12 = 612$

e) $5 \times 34 = 5 \times (30+4) = (5 \times 30) + (5 \times 4) = 150 + 20 = 170$

f) $9 \times 22 = 9 \times (20+2) = (9 \times 20) + (9 \times 2) = 180 + 18 = 198$

4. Rename each product and divide as shown in the example. Copy each exercise on your paper as you did in 3.

Example:

$$28 \div 2 = (20 + 8) \div 2 = (20 \div 2) + (8 \div 2) = 10 + 4 = 14.$$

Part a) is begun for you but it is not finished.

a) $84 \div 4 = (80 + 4) \div 4 = (80 \div 4) + (4 \div 4) = 20 + 1 = 21$

b) $96 \div 3 = (90 + 6) \div 3 = (90 + 3) + (6 \div 3) = 30 + 2 = 32$

c) $369 \div 3 = (300 + 60 + 9) \div 3 = (300 \div 3) + (60 \div 3) + (9 \div 3) = 100 + 20 + 3 = 123$

d) $999 \div 9 = (900 + 90 + 9) \div 3 = (900 + 3) + (90 \div 3) + (9 \div 3) = 300 + 30 + 3 = 333$

5. Write the letter for each part on your paper. Then beside it write the number that you would put in the blank space, or use for the letter in the sentence, so that each of the following will be a true sentence.

a) $6 \times \underline{(9)} = 54$.

b) $8 \times 8 = \underline{(64)}$.

c) $7 \times 9 = \underline{(63)}$.

d) $8 \times \underline{(9)} = 72$.

e) $\underline{(0)} \times 7 = 0$.

f) $9 \times 9 = \underline{(81)}$.

g) $8 \times q = 48$. (6)

h) $7 \times n = 0$. (0)

i) $n \times n = 9$. $(3 \times 3 = 9)$

j) $(n \times n) \times 4 = 36$ $\{(3 \times 3) \times 4 =$

k) $24 \div 6 = q$. (4)

6. Write the letter for each part on your paper. Then beside it write the one of >, <, = that you would put in the blank space so that each of the following will be a true sentence.

a) $7 \times 4 \ \underline{>} \ 9 \times 3$.

b) $9 \times 5 \ \underline{<} \ 6 \times 8$.

c) $94 - 40 \ \underline{=} \ 6 \times 9$.

d) $n \times 4$ ____ $n + 4$. (If $n = 0$ or $n = 1$, then $n \times 4 < n + 4$.
If $n > 1$, then $n \times 4 > n + 4$.)

e) $6 \times 6 \ \underline{>} \ 5 \times 7 \ \underline{>} \ 8 \times 4$.

f) $8 \times n \ \underline{>} \ 7 \times n$.

7. Find the missing number. Write your answer on your paper beside the letter for each part.

a) 36 ÷ 4 = (9).

b) 81 ÷ 9 = (9).

c) 28 ÷ (4) = 7.

d) (72) ÷ 9 = 8.

e) (16) ÷ 4 = 4.

f) 36 ÷ 6 = (6).

8. Are the following statements true? Write yes or no on your paper beside the letter for each part.

a) 7 × 4 = 4 × 7. (yes)

b) 12 - 5 = 5 - 12. (no)

c) 10 - 5 = 5 ÷ 10. (no)

d) 6 + 9 = 9 + 6. (yes)

e) 5 × 34 = 5 × (2 × 17). (yes)

f) (2 × 5) × 3 = 2 × (5 × 3). (yes)

g) 25 × 8 = (10 + 10 + 5) × 8. (yes)

h) 51 × 49 = (50 × 50) - 1. (yes)

i) 80 ÷ 5 = (80 ÷ 10) ÷ 2. (no)

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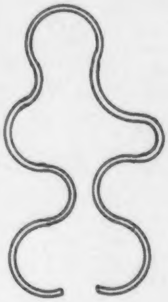
SETS OF POINTS

1. Write the letter for each part on your paper. Then beside the letter write true if the statement is true. If the statement is not true, write false.



a) Space is a set of points. (true)

b) A curve is a set of points. (true)

c) This is a model of a simple closed curve: (true)

d) A ray has one endpoint. (true)

e) A line segment has one endpoint. (false)

f) A line hasn't any endpoints. (true)

g) There is only one plane in space. (false)

h) A plane may contain many lines. (true)

i) Two points in space may be contained in more planes than can be counted. (true)

j) Three points not on a straight line are in one and only one plane. (true)

k) All the radii of a circle have the same length. (true)

l) The union of two rays with a common endpoint is called an angle. (true)

m) A triangle does not contain its angles. (false)

PROPERTIES AND TECHNIQUES OF ADDITION AND SUBTRACTION, II

1. Copy each of these on your paper. Then find the sum in each.

a)	b)	c)	d)	e)
79	327	3287	17289	46060
42	648	4925	42716	25349
36	905	6776	83475	61171
88	36	402	(143,480)	(132,580)
75	(1916)	(15,390)		
(320)				

2. Copy each of these on your paper. Then subtract. After the subtraction, undo each one to show that your answer to the subtraction was correct.

a)	"undo"	b)	"undo"
1636	724	4321	1231
724	912	1231	3090
912	1636	3090	4321

c)	"undo"
1417	519
519	898
898	1417

3. On your paper write a mathematical sentence for each part. Then solve to find the answer to the problem. Write an answer sentence.

a) Don has 600 stamps. He pasted 342 in his album. How many are left to be put in the album? ($600-342=n$
Don has 258 stamps left to be put in the album.)

b) A school's stadium has 12320 seats. The school has sold 6480 tickets for a game. How many tickets are left? ($12,320 - 6,480 = n$ There are 5,840 tickets left to sell.)

c) John wanted to collect 500 shells. He had 188. His uncle gave him 123. How many more did he need to complete his collection? ($188 + 123 + n = 500$
John needs 189 more shells to complete his collection.)

d) Suppose you are going on an automobile trip of 1260 miles. You travel 418 miles the first day and 390 miles the second day. How many miles must you travel on the third day to complete the trip? ($418 + 390 + n = 1260$
You must travel 452 miles to complete the trip.)

e) The earth is 92,900,000 miles from the sun. Mars is 141,000,000 miles from the sun. How much closer to the sun is the earth than Mars is?
($141,000,000 - 92,900,000 = n$ Earth is 48,100,000 miles closer to the sun than Mars.)

TECHNIQUES OF MULTIPLICATION AND DIVISION

1. Study the example in part a). Then copy on your paper the exercises in b), c), d) and multiply as we do in a).

	a)	b)	c)	d)
	38	95	76	85
	24	24	32	79
	32 = 4 × 8	20 = 4 × 5	12 = 2 × 6	45 = 9 × 5
	120 = 4 × 30	360 = 4 × 90	140 = 2 × 70	720 = 9 × 80
	160 = 20 × 8	100 = 20 × 5	180 = 30 × 6	350 = 70 × 5
	600 = 20 × 30	1800 = 20 × 90	2100 = 30 × 70	5600 = 70 × 80
Sum	912 = 24 × 38	2280 = 24 × 95	2,432 = 32 × 76	6,715 = 79 × 85

2. Copy each of these problems in multiplication on your paper and then find the product in each.

a)	b)	c)	d)
432	516	237	489
26	47	69	56
(11,232)	(24,252)	(16,353)	(27,384)

3. The parts of this exercise are division problems. In each one there will be a remainder. Perform each one of the divisions on your paper. Then write a mathematical sentence as in a) to show the remainder. Letter each part as shown here.

a) 621 ÷ 15 Sentence: 621 = (15 × 41) + 6

b) 983 ÷ 24 (983 = (24 × 40) + 23)

c) 671 ÷ 61 (671 = (61 × 11) + 0)

d) 1934 ÷ 21 (1934 = (21 × 92) + 2)

e) 2109 ÷ 9 (2109 = (9 × 234) + 3)

4. A school building has 40 rooms. The school ordered 28 new chairs for each room. When the chairs were delivered there were 1128 chairs. Were there too many or not enough? On your paper show how you would find the answer to the question. ($40 \times 28 = 1120$ $1120 < 1128$
There were too many chairs delivered.)

5. Show on your paper how you find the answers to the questions in this problem. The Parent Teacher Association of a school had 324 members. These were divided into teams of 8 members each.

How many teams could there be with 8 members?
($324 \div 8$. $324 = (8 \times 40) + 4$ There could be 40 teams with 8 members each.)

Were any members of the Association "left over"? (yes)

What is the largest number of teams that could have just 8 members and how many teams would there be that have less than 8 members so that all 324 persons would be in a team? (There could be 40 teams of 8 members each and one team with only 4 members.)

6. Show on your paper how you find the answers to the question in this problem.

There are 16 piles of blocks. In each pile there are 144 blocks. How many blocks are there in the 16 piles?
($16 \times 144 = n$ There are 2304 blocks in the 16 piles.)

How many more blocks would be needed to have 2400 blocks?
($2304 + n = 2400$. We would need 96 more blocks to have 2400 blocks.)

RECOGNITION OF COMMON GEOMETRIC FIGURES

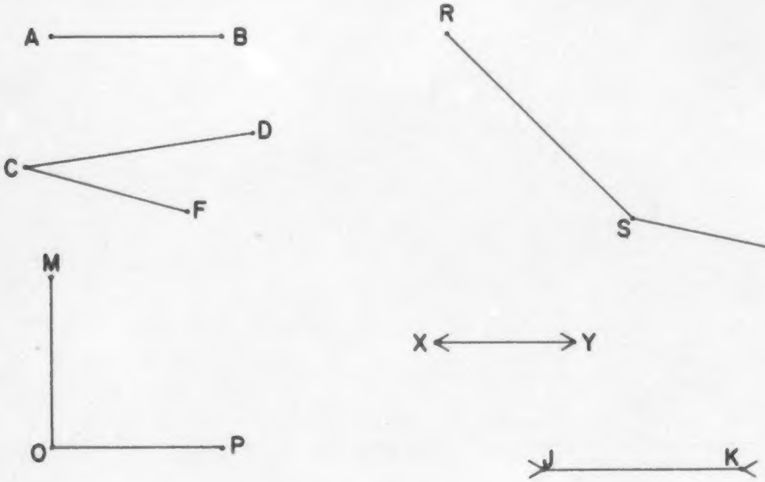
1. Write the letter for each part of this exercise on your paper. Then beside it write the words or word that you would use to fill the blank spaces or space.

   a) A polygon which is the union of three line segments is called a (triangle).

   b) A polygon which is the (union) of (four) line segments is called a quadrilateral.

   c) The endpoints of the line segments in the polygons in a) and b) are called (vertices).

2. Here are some line segments. The segment AB is congruent to some of them. Write the names of the segments to which $\overline{AB}$ is congruent. ($\overline{CF}$, $\overline{OP}$, $\overline{OM}$, $\overline{ST}$)

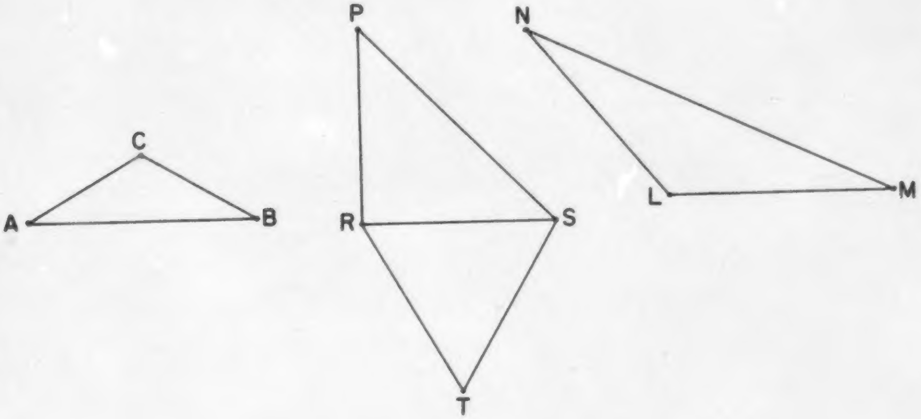


P566

3. A triangle which has at least two sides congruent to each other is called an isosceles triangle. A triangle which has all three sides congruent to each other is called an equilateral triangle. Answer these questions on your paper.

a) Is an equilateral triangle an isosceles triangle? (yes)

b) Are all of the triangles drawn below isosceles triangles? (yes)



c) Write the names of the ones that are equilateral triangles. (ΔRST)

d) Write the names of the ones that are not equilateral triangles. (ΔABC, ΔRSP, ΔLMN)

P567

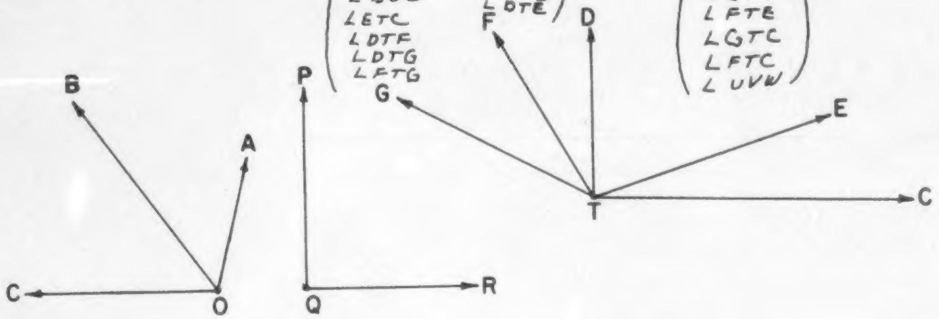
4. Make a model of a right angle by folding a sheet of paper. Now use your model to find which of the angles below are right angles.

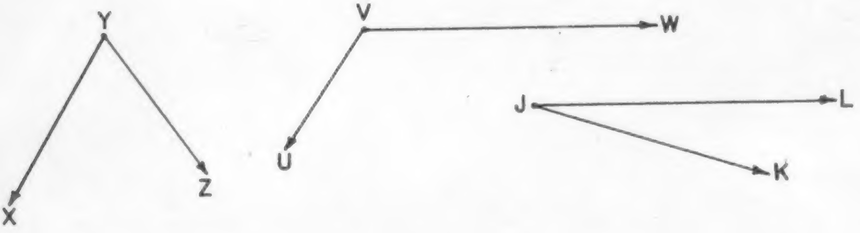
Which angles are less than right angles?

Which angles are greater than right angles?

List the angles on your paper. Put the name of each angle under the proper heading:

Right Angles	Less than a right angle	Greater than a right angle
∠PQR	∠AOB	∠COA
∠DTC	∠BOC	∠GTE
	∠ETC	∠FTE
	∠DTF	∠GTC
	∠DTG	∠FTC
	∠FTG	∠UVW
	∠XYZ	
	∠LKG	
	∠DTE	





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LINEAR MEASUREMENT

In exercises 1, 2, 3, 4 write the letter for each part on your paper. Then beside it write what you would write to fill the blanks.

1. A family drinks 5 quarts of milk each day.

   a) the unit of measure is (quart).

   b) The measure is (5).

   c) The amount of milk is (5 quarts).

2. My automobile weighs 2860 pounds.

   a) The unit of measure is (pound).

   b) The measure is (2860).

   c) The automobile's weight is (2860 pounds).

3. The teacher's desk is 42 inches long.

   a) Its length is (42 inches).

   b) Its measure is (42).

   c) The unit of measure is (inch).

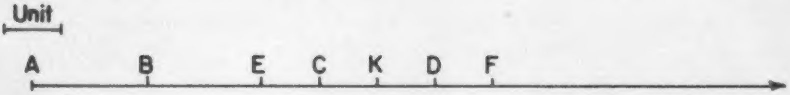
4. A satellite's distance from the earth was 450 miles.

   a) The distance from the earth is (450 miles).

   b) The unit of measure is (mile).

   c) The measure of the distance is (450).

5. The unit to be used in this exercise is shown. Points are named on the ray. Use your compass to find the measure of the segments. On your paper write the measure of the segment beside the letter for each part.



a) m $\overline{AE}$ (4)

b) m $\overline{BE}$ (2)

c) m $\overline{BF}$ (6)

d) m $\overline{AD}$ (7)

e) m $\overline{CF}$ (3)

f) m $\overline{AF}$ (8)

6. Using the unit and the segments in 5 write on your paper what you would write to fill the blank in each part below. May you choose more than one answer for the blank? (yes. Other possible answers are shown below.)

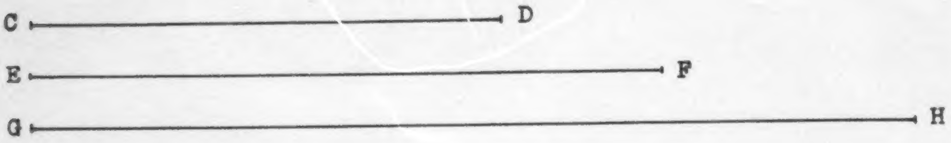
a) m $\underline{\overline{BF}}$ = 6. $\overline{AK}$

b) m $\underline{\overline{BE}}$ = 2. $\overline{AB}$, $\overline{EK}$, $\overline{CD}$, $\overline{KF}$

c) m $\underline{\overline{CF}}$ = 3. $\overline{BC}$, $\overline{ED}$

d) m $\underline{\overline{DF}}$ = 1. $\overline{EC}$, $\overline{CK}$, $\overline{KD}$

7. On your paper make the table like the one below these line segments. Then use your ruler to help you fill in the number which belongs in each blank.



	To the nearest inch	To the nearest half-inch	To the nearest fourth-inch
$\overline{CD}$	(3)	$(2\frac{1}{2})$	$(2\frac{3}{4})$
$\overline{EF}$	(4)	$(3\frac{1}{2})$	$(3\frac{3}{4})$
$\overline{GH}$	(5)	(5)	$(5\frac{1}{4})$

8. On your paper write the letter for each part of this exercise. Then beside it write what you would write to fill in the blanks for each part.

If segments have these lengths, which one is longer? How much longer?

	Which is Longer	How much Longer
a) 23 inches or 1 foot?	(23 inches)	(11 inches)
b) 2 feet or 1 yard?	(1 yard)	(1 foot)
c) 1 ft. 2 in. or 2 ft.?	(2 ft.)	(10 in.)
d) 1 yd. 2 ft. or 6 ft.?	(6 ft.)	(1 ft.)
e) 1 mile or 5300 ft.?	(5300 ft.)	(20 ft.)
f) 16 in. or 2 ft. 2 in.?	(2 ft. 2 in.)	(10 in.)
g) 130 in. or 11 ft. 2 in.?	(11 ft. 2 in.)	(4 in.)

9. Add these measures. Write the answers on your paper beside the letter which names each part. Write each answer 2 ways.

a) 6 yd. 2 ft.
5 yd. 1 ft.
2 yd. 2 ft.
(13 yd. 5 ft.)
(14 yd 2 ft

b) 12 ft. 11 in.
16 ft. 5 in.
24 ft. 8 in.
(52 ft 24 in. or 54 ft.)

c) 2 yd. 2 ft. 3 in.
6 yd. 1 ft. 10 in.
5 yd. 1 ft. 11 in.
(13 yd. 4 ft. 24 in. or 15 yd)

10. Subtract these measures. Write the answers on your paper beside the letter which names each part.

a) 5 yd. 2 ft.
3 yd. 1 ft.
(8 yd. 3 ft or 9 yd.)

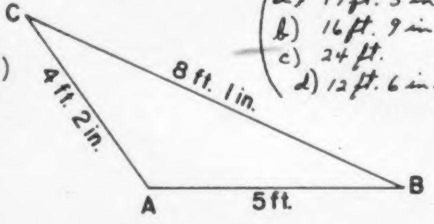
b) 6 yd. 2 ft. 11 in.
4 yd. 2 ft. 5 in.
10 yd. 4 ft. 17 in

c) 7 ft. 8 in.
4 ft. 10 in.

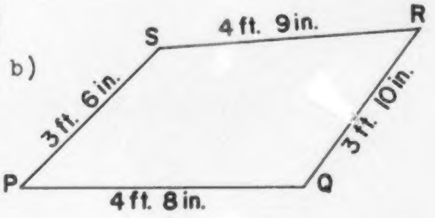
11. Find the perimeter of each polygon. On your paper write the answer beside the letter which names each polygon.

(a) 17 ft. 3 in.
b) 16 ft. 9 in.
c) 24 ft.
d) 12 ft. 6 in.)

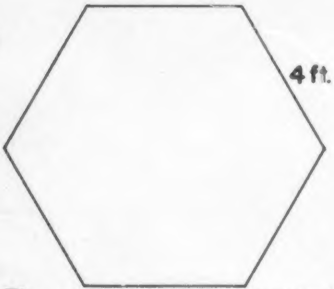
a)


C
8 ft. 1 in.
4 ft. 2 in.
A
5 ft.
B

b)

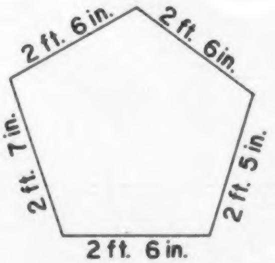

S
4 ft. 9 in.
R
3 ft. 6 in.
3 ft. 10 in.
P
4 ft. 8 in.
Q

c)



The segments are congruent

d)


2 ft. 6 in.
2 ft. 6 in.
2 ft. 7 in.
2 ft. 5 in.
2 ft. 6 in.

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EXTENDING SYSTEMS OF NUMERATION

1. On your paper write the word names for the following numbers. Letter the parts of this exercise as they are lettered here.

   a) 2,536 (two thousand five hundred thirty-six)
   b) 45,269 (forty-five thousand two hundred sixty-nine)
   c) 40,204 (forty thousand two hundred four)
   d) 60,066 (sixty thousand sixty-six)
   e) 66,066 (sixty-six thousand sixty-six)
   f) 66,000 (sixty-six thousand)
   g) 66,606 (sixty-six thousand six hundred six)
   h) 124,301 (one hundred twenty-four thousand three hundred one)

2. On your paper write the numerals for the following numbers. Letter the parts of this exercise as they are lettered here.

   a) Two thousand five hundred twenty. (2,520)
   b) Three thousand three hundred thirty. (3,330)
   c) Fifty-five thousand five hundred fifty-five. (55,555)
   d) Nine thousand seventy-six. (9,076)
   e) One thousand seven hundred seventy-six. (1,776)
   f) Twenty thousand two hundred two. (20,202)
   g) One thousand two. (1,002)
   h) Eleven thousand one hundred eleven. (11,111)

FACTORS AND PRIMES

1. On your paper write the letter for each part of this exercise. Then beside it complete each statement. Complete the statement so that each number is a product of 3 factors. Part a) is done for you.

   a) $24 = 2 \times 3 \times 4$
   b) $18 =$ $(2 \times 3 \times 3)$
   c) $36 =$ $(4 \times 3 \times 3$ or $6 \times 2 \times 3$ or $2 \times 2 \times 9)$
   d) $12 =$ $(2 \times 2 \times 3)$
   e) $8 =$ $(2 \times 2 \times 2)$

2. For each set of 3 numbers write on your paper the smallest number which has each of the 3 numbers as a factor. Letter each part as lettered here.

   a) 2, 5, 7 (70)
   b) 2, 3, 4 (12)
   c) 5, 7, 1 (35)
   d) 4, 6, 8 (24)
   e) 2, 4, 8 (8)
   f) 3, 6, 9 (18)

3. Some of the following numbers are prime numbers. Write the prime numbers on your paper.

   a) 27
   b) 31 (prime)
   c) 55
   d) 53 (prime)
   e) 310
   f) 143
   g) 37 (prime)
   h) 101 (prime)

4. Find two different prime factors of each of these numbers. Write the prime factors on your paper. Letter the parts as they are lettered here.

a) 785 (5, 157)
b) 3,042 (2, 3)
c) 5,055 (5, 3)
d) 6,060 (2, 3, 5)
e) 4,314 (2, 3)

5. On your paper write the set of all factors of each number. Letter the parts as they are lettered here.

a) 96 (1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96)
b) 225 (1, 3, 5, 9, 15, 25, 45, 75, 225)
c) 363 (1, 3, 11, 33, 121, 363)
d) 189 (1, 3, 7, 9, 21, 27, 63, 189)

6. Find the greatest common factor of the following pairs of numbers. Use same letters for your answers that are used here.

a) 90, 84 (6)
b) 90, 50 (10)
c) 72, 60 (12)
d) 48, 30 (6)
e) 12, 9 (3)

7. Find the lowest common multiple of the numbers in each set. Write the lowest common multiple on your paper. Letter the parts as lettered here.

a) 6, 8, 12 (24)
b) 6, 15 (30)
c) 3, 5, 9, 15 (45)
d) 4, 8, 12 (24)
e) 3, 6, 5, 9 (90)
f) 3, 4, 5, 10, 12 (60)

8. We say that a number is "factored completely" if it is the product of numbers which are all prime numbers. Factor completely each of the following numbers and write them on your paper as in a) which is done for you.

a) 63 = 3 × 3 × 7
b) 126 = (2 × 3 × 3 × 7)
c) 49 = (7 × 7)
d) 98 = (2 × 7 × 7)
e) 35 = (5 × 7)
f) 105 = (3 × 5 × 7)
g) 45 = (3 × 3 × 5)
h) 135 = (3 × 3 × 3 × 5)
i) 1001 = (7 × 11 × 13)

P576

EXTENDING MULTIPLICATION AND DIVISION

1. Find the product for each product expression.

a) $3 \times 46 = n$ (138)
b) $7 \times 83 = n$ (581)
c) $5 \times 125 = n$ (625)
d) $6 \times 321 = n$ (1926)
e) $4 \times 1269 = n$ (5076)
f) $12 \times 34 = n$ (408)
g) $23 \times 67 = n$ (1541)
h) $52 \times 48 = n$ (2496)
i) $76 \times 94 = n$ (7144)
j) $38 \times 83 = n$ (3154)

2. Find the number represented by n to make each sentence true.

a) $7 \times n = 3402$ (486)
b) $n \times 21 = 966$ (46)
c) $n \times 18 = 486$ (27)
d) $24 \times n = 1536$ (64)
e) $43 \times 267 = n$ (11,481)
f) $58 \times 131 = n$ (7598)
g) $n \times 81 = 8667$ (107)
h) $14 \times 463 = n$ (6482)
i) $37 \times 1249 = n$ (46,213)
j) $n \times 125 = 9250$ (74)

3. Find the numbers represented by n and r for each of the following so that they are true mathematical sentences.

a) $487 = (n \times 43) + r$ $[(11 \times 43) + 14]$
b) $396 = (n \times 61) + r$ $[(6 \times 61) + 30]$
c) $1292 = (34 \times n) + r$ $[(34 \times 38) + 0]$
d) $3415 = (53 \times n) + r$ $[(53 \times 64) + 23]$
e) $8645 = (n \times 65) + r$ $[(133 \times 65) + 0]$
f) $9772 = (n \times 73) + r$ $[(133 \times 73) + 63]$
g) $12{,}443 = (n \times 120) + r$ $[(103 \times 120) + 83]$
h) $24{,}811 = (151 \times n) + r$ $[(151 \times 164) + 47]$

In which of these does n represent a factor of the number given? (c and e)

4. For each sentence, find the number that n must represent to make the sentence true.

a) $1541 = (n \times 37) + 24$ $(n = 41)$

b) $3255 = (n \times 24) + 15$ $(n = 135)$

c) $6189 = (73 \times n) + 57$ $(n = 84)$

d) $9888 = (n \times 44) + 32$ $(n = 224)$

5. Find n in each of these.

a) $(5 \times 7) + (6 \times 7) = n$ $(n = 77)$

b) $(10 \times 15) + (10 \times 2) = n$ $(n = 170)$

c) $(14 \times 6) + (3 \times 6) = n$ $(n = 102)$

d) $(8 + 2) \times 5 = n$ $(n = 50)$

e) $7 \times (100 + 6) = n$ $(n = 742)$

f) $(2 + 4 + 3) \times 5 = n$ $(n = 45)$

6. Express 207 as the product of two factors, one of which is 23. $(207 = 23 \times 9)$

7. What is the product of 21 and the next odd number? $(21 \times 23 = 483)$

8. What is the product of 21 and the next even number? $(21 \times 22 = 462)$

9. Is 360 a multiple of 45? (yes $360 = 45 \times 8$)

10. Is 13 a factor of 101? (No)

Using Multiplication and Division

11. The width of a playground is 55 yards. Its length is 120 yards. Find the area of the playground. (Area $= 55 \times 120$ The area of the playground is 6600 square yards.)

12. Tim sold 45 papers each day. How many papers did he sell in the month of May? ($45 \times 31 = n$ Tim sold 1395 papers in the month of May.)

13. Three girls divided 47 pictures equally among them. How many pictures did each girl get? (They cannot be divided equally. Each girl gets 15 pictures and there are two pictures left over.) How many more pictures do they need so each girl will have 25 pictures? ($47 + n = 75$ They will need 28 more pictures.)

14. There were 79 cookies on a tray. How many dozen cookies were there on the tray? ($79 \div 12 = n$ There were 6 dozen and 7 cookies left over.)

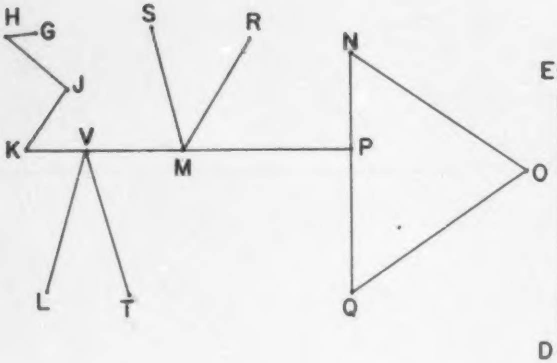
15. On another tray there were 5 times as many cookies as on the tray in Problem 14. How many cookies are on that tray? ($5 \times 79 = n$ There were 395 cookies on that tray.) How many dozen cookies are on that tray? ($395 \div 12 = n$ There are 32 dozen cookies and eleven cookies are left over.)

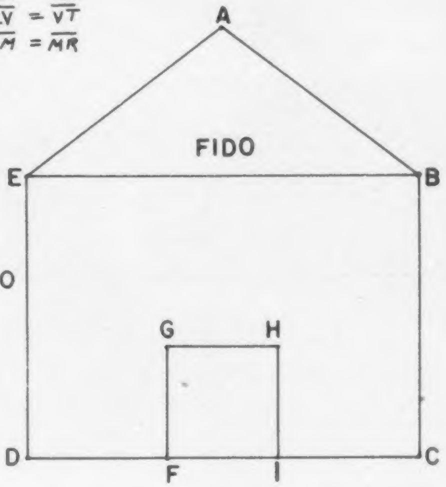
16. How many dozen cookies are on both trays?
($79 + 395 = n$ There are 474 cookies on both trays.
$474 \div 12 = s$ There are 39 dozen cookies on both trays.)

CONGRUENCE OF COMMON GEOMETRIC FIGURES

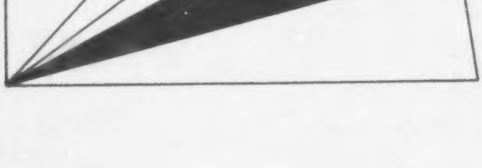
1. Find the congruent segments in each figure. Trace the segments on a sheet of thin paper or use your compass to help you decide.

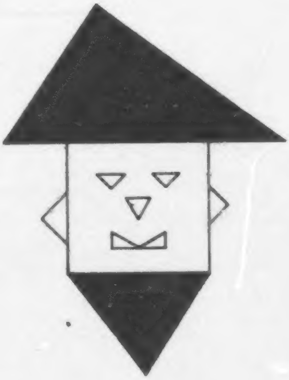
$\overline{ON} = \overline{OQ}$, $\overline{GH} = \overline{FI}$
$\overline{EA} = \overline{AB}$, $\overline{GF} = \overline{HI}$
$\overline{ED} = \overline{BC}$, $\overline{LV} = \overline{VT}$
$\overline{EB} = \overline{DC}$, $\overline{SM} = \overline{MR}$



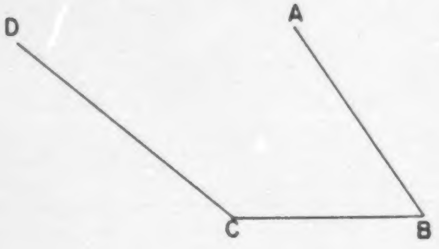


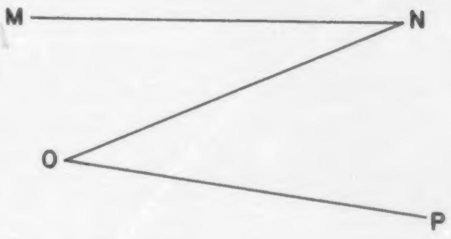
2. Use your compass and straightedge to copy each of the triangles whose interior is shaded.



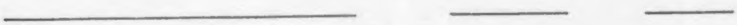


3. Use your compass and straightedge to draw a triangle using the given segments.

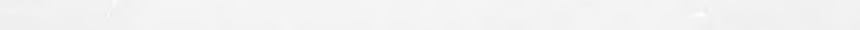




4. Can you construct a triangle, using these three line segments? (No.) Why? (Because the sum of the measures of the two short sides is less than the measure of the longest side.)



5. What would be true about the triangles constructed from these three line segments? (It would be an equilateral triangle.)

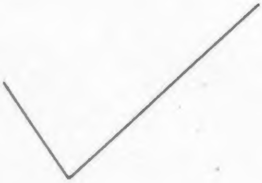


What does this statement mean? "Three sides determine a triangle."

6. Use your compass and straightedge to copy the check mark made by Bill's teacher.

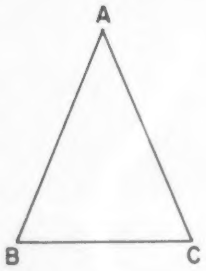
Bill's Problem

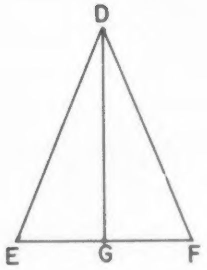
$$\begin{array}{r} 85 \\ \underline{\times\ 39} \\ 775 \\ \underline{255} \\ 2325 \end{array}$$



Does the check mark mean that Bill's problem has the right answer or the wrong answer? (The check mark means Bill's problem has the wrong answer.)

7.





Write, in words, these mathematical sentences.

a) $\triangle ABC \cong \triangle DEF$ and $\overline{AB} > \overline{BC}$. (Triangle ABC is congruent to triangle DEF and line segment AB is greater than line segment BC.)

b) $\triangle ABC \neq \triangle DEF$ and $\overline{EG} < \overline{DF}$.
Triangle ABC is not congruent to triangle DEF and line segment EG is less than line segment DF.

ADDITION AND SUBTRACTION OF RATIONAL NUMBERS

1. On your paper write the letter for each part of this exercise. Then beside it write the one of >, =, < that you would use to fill the blank so each of the following would be a true mathematical sentence.

a) $\frac{1}{2}\ \underline{>}\ \frac{1}{3}$

b) $\frac{1}{4}\ \underline{<}\ \frac{1}{3}$

c) $\frac{2}{3}\ \underline{<}\ \frac{3}{4}$

d) $\frac{7}{4}\ \underline{<}\ \frac{6}{3}$

e) $\frac{3}{6}\ \underline{>}\ \frac{2}{5}$

f) $\frac{0}{4}\ \underline{=}\ 0$

g) $\frac{3}{6}\ \underline{=}\ \frac{17}{34}$

h) $\frac{1}{5}\ \underline{>}\ \frac{12}{70}$

i) $\frac{4}{16}\ \underline{=}\ \frac{1}{4}$

j) $\frac{5}{9}\ \underline{<}\ \frac{11}{18}$

k) $\frac{3}{7}\ \underline{<}\ \frac{3}{6}$

l) $\frac{1}{3}\ \underline{<}\ \frac{2}{4}$

2. On your paper write the letter for each part of this exercise. Then beside it write the lowest common denominator for the rational numbers in the set.

a) $\frac{3}{4}, \frac{1}{5}, \frac{5}{6}$ (60)

b) $\frac{7}{8}, \frac{3}{4}, \frac{1}{6}$ (24)

c) $\frac{7}{10}, \frac{5}{12}, \frac{1}{6}$ (60)

d) $\frac{5}{12}, \frac{3}{16}, \frac{2}{3}$ (48)

3. Pick out the rational number in each set which is the largest. Write it on your paper beside the letter for that set.

a) $\frac{3}{4}, \frac{4}{5}, \frac{6}{10}$ $\left(\frac{4}{5}\right)$

b) $\frac{1}{7}, \frac{1}{8},$ $\left(\frac{1}{7}\right)$

c) $\frac{1}{9}, \frac{1}{10}, \frac{1}{11}$ $\left(\frac{1}{9}\right)$

d) $\frac{5}{8}, \frac{2}{3}, \frac{3}{4}$ $\left(\frac{3}{4}\right)$

4. On your paper write the letter for each part of this exercise. Then beside it write the number you would use for n to make the mathematical sentence true.

a) $n = \frac{5}{8} + \frac{7}{8}$ $(n = \frac{12}{8})$

b) $\frac{2}{3} + \frac{11}{12} = n$ $(n = \frac{19}{12})$

c) $\frac{3}{4} + \frac{27}{20} = n$ $(n = \frac{42}{20})$

d) $\frac{3}{5} + \frac{1}{5} = n$ $(n = \frac{4}{5})$

e) $n = \frac{3}{5} + \frac{5}{6}$ $(n = \frac{43}{30})$

f) $\frac{1}{2} + \frac{2}{3} = n$ $(n = \frac{7}{6})$

5. On your paper write the letter for each part of this exercise. Then beside it write the number you would use for x to make the mathematical sentence true.

a) $x = \frac{7}{8} - \frac{1}{2}$ $(x = \frac{3}{8})$

b) $\frac{11}{12} - \frac{5}{6} = x$ $(x = \frac{1}{12})$

c) $\frac{1}{2} + \frac{2}{5} = x$ $(x = \frac{9}{10})$

d) $\frac{3}{4} - \frac{1}{6} = x$ $(x = \frac{7}{12})$

e) $\frac{7}{8} - \frac{9}{16} = x$ $(x = \frac{5}{16})$

f) $x = \frac{2}{5} - \frac{1}{4}$ $(x = \frac{3}{20})$

6. On your paper write the letter for each part of this exercise. Then beside it write the number you would use for p.

a) $\frac{1}{3} + \frac{1}{2} + \frac{1}{4} = p$ $(p = \frac{13}{12})$

b) $\frac{3}{5} + \frac{3}{4} + \frac{1}{2} = p$ $(p = \frac{37}{20})$

c) $\frac{5}{8} + \frac{2}{3} + \frac{1}{2} = p$ $(p = \frac{43}{24})$

d) $\frac{3}{8} + \frac{1}{4} + \frac{5}{6} = p$ $(p = \frac{35}{24})$

7. Copy each part of this exercise on your paper. Then fill each blank with + or - so that a), b), c), and d) will be true mathematical sentences.

a) $\frac{4}{3} \underline{(+)} \frac{2}{3} \underline{(-)} \frac{5}{3} = \frac{1}{3}$

b) $\frac{12}{7} \underline{(+)} \frac{3}{7} \underline{(+)} \frac{8}{7} = \frac{23}{7}$

c) $\frac{9}{8} \underline{(+)} \frac{7}{8} \underline{(-)} \frac{12}{8} = \frac{4}{8}$

d) $\frac{10}{3} \underline{(-)} \frac{5}{4} \underline{(+)} \frac{1}{6} = \frac{9}{4}$

8. On your paper write the letter for each part of this exercise. Then beside it write the number for n so that the sentence will be true.

a) $(4\frac{1}{6} + 3\frac{2}{6}) + 5\frac{2}{6} = n$ $(12\frac{5}{6})$

b) $(6\frac{2}{8} + \frac{3}{8}) + n = 10\frac{7}{8}$ $(4\frac{2}{8}$ or $4\frac{1}{4})$

c) $(3\frac{3}{4} + 2\frac{1}{2}) + 6\frac{1}{3} = n$ $(11\frac{19}{12}$ or $12\frac{7}{12})$

d) $(8\frac{2}{5} - 2\frac{7}{10}) + 3\frac{1}{4} = n$ $(8\frac{19}{20})$

9. On your paper write the letter for each part of this exercise. Then beside it write the number for n so that the sentence will be true.

a) $(8.97 - 4.31) + n = 11.89$ $(n = 7.23)$

b) $3.24 + 3.56 + 4.16 = n$ $(10.96 = n)$

c) $7.88 + 5.31 + 6.54 = n$ $(19.73 = n)$

d) $6 + .3 + n = 6.36$ $(n = 0.06)$

e) $8.34 - 4.83 = n$ $(3.51 = n)$

f) $n = 9.34 - 5.89$ $(n = 3.45)$

MEASUREMENT OF ANGLES

1. On your paper write the letter for each part of this exercise. Beside the letter write the word true if the statement is true. If the statement is false, write the word false.

   a) A measure of an angle is a number. (true)

   b) The unit used for measuring angles is an angle. (true)

   c) A measure is not a number. (false)

   d) The measure is not accurate, but is only approximately so. (true)

   e) The instrument used for angle measure is a protractor. (true)

   f) The sides of an angle are rays. (true)

   g) The common endpoint of the two rays forming an angle is the vertex. (true)

   h) The measure of the angle depends upon the lengths of the rays. (false)

   i) Every angle has one ray drawn horizontally. (false)

   j) The standard unit of angle measure is the degree. (true)

   k) The measure in degrees of each angle of an equilateral triangle is 60. (true)

   l) Angles may be of the same measure but be different in positions. (true)

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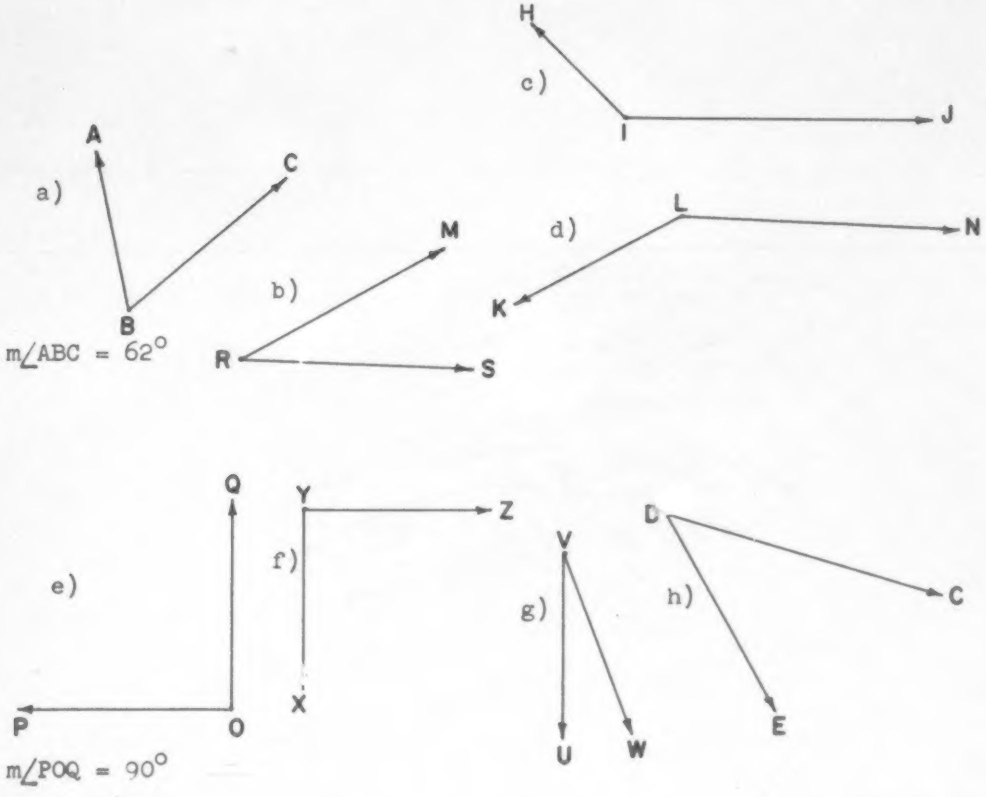
2. In each figure below the measure in degrees of certain angles are shown. On your paper write the letter that names each angle whose measure in degrees is not shown. Then beside the letter write the measure of the angle in degrees.


a)
A
C
B
b)
M
R
S
c)
H
I
J
d)
L
N
K
e)
Q
P
O
f)
Y
Z
X
g)
V
U
W
h)
D
C
E

m∠ABC = 62°

m∠POQ = 90°

b. m∠MRS = 31°
c. m∠HIJ = 135°
d. m∠KLN = 150°
e. m∠POQ = 90°
f. m∠XYZ = 90°
g. m∠UVW = 21°
h. m∠EDC = 44°

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AREA

1. On your paper write the letter for each part of this exercise. Then beside it write the word, or words, that you would use to fill the blanks.

a) A (square unit) is used as a unit for measuring plane regions.

b) A simple closed curve separates a plane into (three) sets of points.

c) The union of a simple closed curve and its interior is called a (plane region).

d) To measure area of a region we need a unit of (measure).

e) One standard unit of area is a square region with 1-inch sides; this unit is called the (square inch).

f) An area of 1 square yard is the same as an area of (nine) square feet.

g) 4 square yards = (36) square feet.

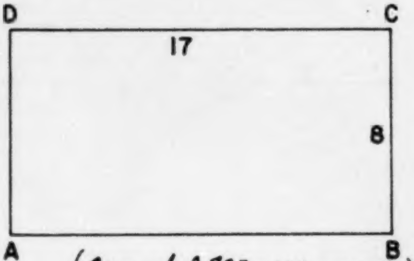
h) 2 square feet = (288) square inches.

i) 7 square feet and 24 square inches = (1032) square inches.

j) 2000 square inches = (1) square yards and (704) square inches.

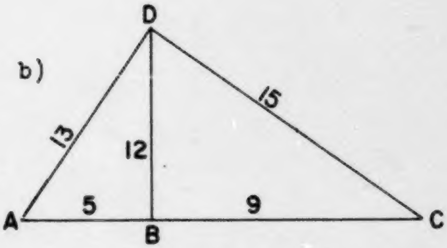
2. The polygons shown below are either rectangles or triangles. The numbers are the measures. Find the measure in square units of the area of each rectangular and triangular region and write it on your paper beside the name of the rectangle or triangle.

a)



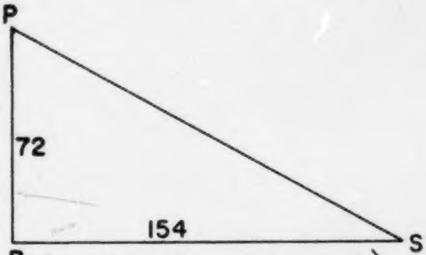
(Area of ABCD = 136 sq units)

b)



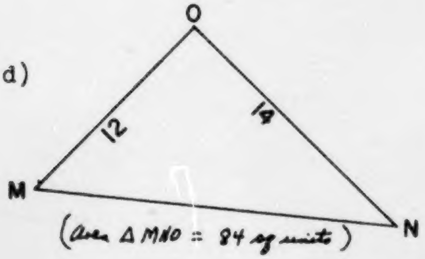
Area of △ABD = (30 sq unit)
Area of △CBD = (54 sq unit)
Area of △ACD = (84 sq unit)

c)



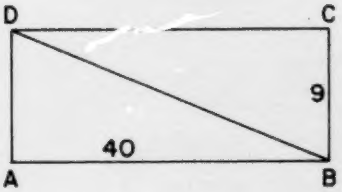
(area of △ RSP = 5544 sq units)

d)



(area △ MNO = 84 sq units)

e)



Area of △ABD = (180 sq unit)
Area of △BCD = (180 sq unit)
Area of ABCD = (360 sq unit)

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